The Methods for Solving the Problems of the Diffraction of Electromagnetic and Acoustic Waves Using the Information on Analytical Properties of the Scattered Fields*

A. G. KYURKCHAN[a], A. I. SUKOV[b] and A. I. KLEEV[c]

[a]Moscow Technical University of Communication and Information Science
Aviamotornaya 8a, Moscow, 105855, Russia
Home address: Staromonetnii by-street 24, apt 15, Moscow, 109180, Russia

[b]Mathematical Department, Moscow State Technological University STANKIN
Vadkowski per. 3A, Moscow 101472, Russia

[c]P.Kapitza Institute for Physical Problems
Kosygina 2, Moscow, 117334, Russia. Fax: 7(095) 938-2030, E-mail: kleev@magnit.msk.su

Abstract — **In spite of impressive achievements in computing technology, the study of numerical-analytical methods for solving boundary value problems in physical diffraction theory continues to be of paramount importance. This is a consequence of the fact that using such methods, we obtain a significant portion of information on solution properties at the stage of analytical investigation of the problem. This information is introduced in the problem algorithm, thus ensuring its adequacy to the problem and, consequently, its effectiveness, stability, high-performance capability etc. Conversely, ignoring the analytical solution properties results, at best, in low speed of convergence and may involve the complete loss of stability, i.e. "computation catastrophe". In this study we examine some approaches to solving boundary problems in electrodynamics and acoustics using *a-priory* information on analytical solution properties.**

1. Methods employing wave harmonics series expansions

Numerical realizations of several widely practiced methods, such as the method of separation of variables, zero-field approach [1], T-matrix approach [2] etc., often involve the field representation in the form of wave harmonics (multipole) series expansions.

Scalar single scatterer diffraction problems use the expansions in the form

$$u^e(\vec{r}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} h_n^{(2)}(kr) P_n^m(\cos\theta) \exp(im\varphi) \quad (1)$$

for representation of the field external to the scatterer (in the region D_e), or in the form

$$u^i(\vec{r}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_{nm} j_n(kr) P_n^m(\cos\theta) \exp(im\varphi) \quad (2)$$

for the field in the internal region (D_i), where j_n, $h_n^{(2)}$ are the spherical Bessel functions; P_n^m is the adjoined Legendre function; $k = 2\pi/\lambda$; λ is the wavelength; dependence on time is taken to be $\exp(ickt)$, where c is the light speed.

Vector problems with expansions in the form (1) and (2) use the vector harmonics

$$\vec{M}_{mn} = \nabla \times (\vec{r}\psi_{mn}),\ \vec{N}_{mn} = \frac{1}{k}\nabla \times \vec{M}_{mn}, \quad (3)$$

where

$$\psi_{mn} = Z_n(kr) P_n^m(\cos\theta)\exp(im\varphi), \quad (4)$$

where Z_n is the corresponding spherical Bessel function. Besides, the vector problems, especially those pertaining to the antenna theory, widely use the Wilcox expansion [3, 4]:

$$\vec{E}(\vec{r}) = \frac{\exp(-ikr)}{kr} \sum_{n=0}^{\infty} \frac{\vec{E}_n(\theta,\varphi)}{(kr)^n}, \quad (5)$$

where the coefficients $\vec{E}_n$ are determined through the use of recurrences from the function $\vec{E}_0(\theta,\varphi)$ representing the wave field pattern [3].

*This work is supported by the Russian Found of Fundamental Researches (grant number 94-02-03122-a)

The expansion (5) is a power series. The representations (1), (2) (and their analogues in the vector case) also, in essence (in principal parts) are power series. Consequently, the existence regions for such representations are

$$r > r_e,$$

for the external problems, and

$$r < r_i,$$

for internal problems, where r_e is the distance to the farthest from the origin of coordinate singularity of the external diffraction field analytical continuation into the scatterer, and r_i is the distance to the nearest singular point of the internal field analytical continuation into the external region [4, 5].

In problems of wave scattering by periodic surfaces, the plane wave series expansions of the form

$$u^{e,i}(x,y) = \sum_{m=-\infty}^{\infty} R_m^{e,i} \exp(-ixw_m \mp iyv_m) \qquad (6)$$

are widely used, where

$$\begin{aligned} w_m &= 2\pi m/b + k\sin\theta, \\ v_m &= \sqrt{k^2 - w_m^2},\ \mathrm{Im} v_m \le 0, \end{aligned}$$

b — the period of the scattering surface S : $y = f(x)$, and θ — the plane wave angle of incidence. In (6), the upper sign corresponds to the representation of the field u^e over the surface S, and the lower sign corresponds to the representation of the field u^i under the surface. The series for u^e converges when

$$y > y_e,$$

and for u^i converges when

$$y < y_i,$$

where y_e is the ordinate of the farthest (along y-axes) singularity of the analytical continuation of the field u^e under the surface S, and y_i is the ordinate of the nearest singular point of the field u_i continued into the region over the surface S [6]. Representations of the type of (6) are also valid for three-dimensional problems. It is significant that quantities r_e, r_i, y_e, y_i can be determined prior to solving a boundary problem [7].

As a rule, the numerical realization of the methods using the above-mentioned representations offers a correct stable algorithm only when there holds so-called Rayleigh hypothesis [8] postulating the continuity of corresponding representations up to the boundary of the S-region wherein the solution is sought, though in some methods (for example, the Waterman method [1]) the Rayleigh hypothesis obviously is not used. Recently, the methods for solving the external problems in electrodynamics and acoustics have been reported, in which the representations of the form (1) are used. These methods result in correct algorithms achieved under considerably less rigorous restrictions on the S-boundary geometry than those satisfying the Raylegh hypothesis [9–11].

2. Method of pattern equations

This method [9–11] may be outlined using the scalar problem in the theory of diffraction by a bounded scatterer as an example. Let us denote the source (primary) field by u^0, the diffraction (secondary) field by u^1 and examine the uniform Dirichlet boundary problem

$$(u^0 + u^1)\big|_S = 0 \qquad (7)$$

for the Helmholtz equation. S is the scatterer surface. For the scattering pattern $g(\theta, \varphi)$, there is the following representation [5]:

$$\begin{aligned} g(\theta,\varphi) = \int_0^{2\pi}\!\!\int_0^{\pi} v(\theta',\varphi') \exp\{ik\rho(\theta',\varphi') \times \\ \times [\sin\theta \sin\theta' \cos(\varphi' - \varphi) + \cos\theta' \cos\theta]\} d\theta' d\varphi', \quad (8) \end{aligned}$$

it being known that

$$u^1(r,\theta,\varphi) = \frac{\exp(-ikr)}{kr} g(\theta,\varphi) + O\left(\frac{1}{(kr)^2}\right).$$

In the relation (8)

$$v(\theta,\varphi) = v^0(\theta,\varphi) + v^1(\theta,\varphi), \qquad (9)$$

and

$$\begin{aligned} v^p(\theta,\varphi) = \frac{k}{4\pi} \Bigg[\rho^2(\theta,\varphi) \sin\theta \frac{\partial u^p}{\partial r} + \rho'_\theta \sin\theta \frac{\partial u^p}{\partial \theta} + \\ + \frac{\rho'_\varphi}{\sin\theta} \frac{\partial u^p}{\partial \varphi} \Bigg]\Bigg|_{r=\rho(\theta,\varphi)}, \ p = 0, 1, \quad (10) \end{aligned}$$

$r = \rho(\theta, \varphi)$ is the equation of the surface S for spherical coordinate system.

Everywhere in $\mathcal{R}^3 \setminus \overline{\mathcal{E}}$, where $\overline{\mathcal{E}}$ is the convex envelope of singularities of the field u^1, the diffraction field can be represented by the integral in the form [11]:

$$u^1(r,\theta,\varphi) = \frac{\exp(-i\pi/4)}{\sqrt{2\pi kr}} \times$$
$$\times \int_{-\pi/2-i\infty}^{\pi/2+i\infty} \exp(-ikr\cos\psi)\hat{g}(\theta,\varphi;\psi)d\psi, \quad (11)$$

where

$$\hat{g}(\theta,\varphi;\psi) = \exp\left[i\psi(B+\tfrac{1}{2})\right] g(\theta,\varphi),$$
$$\hat{g}(\theta,\varphi;0) = g(\theta,\varphi), \quad (12)$$

B is the operator determined from the expression [12]

$$-B(B+1) = D \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \\ +\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \quad (13)$$

Using (8)–(12), we can obtain the following integral-operator equation with respect to $g(\theta,\varphi)$ [11]

$$g(\alpha,\beta) = g^0(\alpha,\beta) + \sqrt{\frac{\pi}{2}}\frac{\exp(-i\pi/4)}{4\pi^2} \times$$
$$\int_0^{2\pi}\int_0^{\pi}\int_{-\pi/2-i\infty}^{\pi/2+i\infty} \frac{1}{\sqrt{k\rho(\theta,\varphi)}} \times$$
$$\left[(ik^2\cos\psi + \frac{k}{2\rho(\theta,\varphi)})\hat{g}(\theta,\varphi;\psi)\rho^2(\theta,\varphi)\sin\theta + \right.$$
$$\left. + \hat{g}'_\theta(\theta,\varphi;\psi)k\rho'_\theta\sin\theta + \frac{k\rho'_\varphi}{\sin\theta}\hat{g}'_\varphi(\theta,\varphi;\psi)\right] \times$$
$$\times \exp\{-ik\rho(\theta,\varphi)[\cos\psi - \sin\theta\sin\alpha\cos(\varphi-\beta) -$$
$$-\cos\theta\cos\alpha]\}d\psi d\theta d\varphi \quad (14)$$

where $g^0(\alpha,\beta)$ is the function related to the transformation (8), on the right of which the function $\vartheta^0(\theta,\varphi)$ should be in place of $\vartheta(\theta,\varphi)$. Eq. (14) is obtained on the assumption that $\overline{\mathcal{E}} \in \overline{D}$, where D is the region inside S. The scatterers meeting the above condition are termed *weakly nonconvex*. In particular, all convex bodies are of this sort. Substituting in (14) series of the type of

$$g(\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_{nm}P_n^m(\cos\theta)\exp(im\varphi) \quad (15)$$

we obtain the following algebraic system for coefficient a_{nm}

$$a_{nm} = a^0_{nm} + \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\nu} G_{nm,\nu\mu}a_{\nu\mu} \quad (16)$$

where

$$a^0_{nm} = i^n(2n+1)\frac{(n-m)!}{(n+m)!} \times$$
$$\times \int_0^{2\pi}\int_0^{\pi} v^0(\theta,\varphi)j_n(k\rho(\theta,\varphi)) \times$$
$$\times P_n^m(\cos\theta)\exp(-im\varphi)d\theta d\varphi$$

(see (9) and (10)), and

$$G_{nm,\nu\mu} = i^{n-\nu}(2n+1)\frac{(n-m)!}{(n+m)!}\frac{i}{4\pi} \times$$
$$\times \int_0^{2\pi}\int_0^{\pi}\{k^2\rho^2(\theta,\varphi)h_\nu^{(2)\prime}(k\rho(\theta,\varphi)) \times$$
$$\times P_\nu^\mu(\cos\theta)\sin\theta - k\rho'_\theta h_\nu^{(2)}(k\rho) \times$$
$$\times [\frac{d}{d\theta}P_\nu^\mu(\cos\theta)]\sin\theta -$$
$$-i\mu\frac{k\rho'_\varphi}{\sin\theta}h_\nu^{(2)}(k\rho)P_\nu^\mu(\cos\theta)\}j_n(k\rho) \times$$
$$\times P_n^m(\cos\theta)\exp(i(\mu-m)\varphi)d\theta d\varphi$$

A significant advantage of the system (16) lies in the fact that its matrix elements are obtained through a double integral instead of a four-fold one being typical for some methods of considerable current use, such as the moment method, the current integral equations etc. For a body of rotation, i.e. when $\rho(\theta,\varphi) = \rho(\theta)$, the system (16) simplifies significantly; in particular, its matrix elements are now obtained through single integrals [11]. Substituting in (16) unknown coefficients in the form

$$a_{nm} = \frac{\sigma_1^n}{n!}x_{nm}$$

we obtain a new algebraic system related to the unknown quantities x_{nm}

$$x_{nm} = x^0_{nm} + \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} g_{nm,\nu\mu}x_{\nu\mu} \quad (17)$$

where

$$x^0_{nm} = \frac{n!}{\sigma_1^n}a^0_{nm}, \; g_{nm,\nu\mu} = G_{nm,\nu\mu}\frac{n!}{\nu!}\sigma_1^{\nu-n}.$$

By analogy with [9], it can be shown that the system (17) is solvable if we use the reduction method with the proviso that

$$\sigma_2 > \sigma_1, \quad (18)$$

where $\sigma_1 = kr_e/2$, $\sigma_2 = kr_i/2$. The criterion (18) is invariant to the kind of the boundary condition on the scatterer surface S. Specifically, in lieu of the condition (7), the conjugation boundary conditions can be met on S (continuity for u and $\partial u/\partial n$, or, in the electromagnetic case, the tangent components of vectors $\vec{E}$ and $\vec{H}$).

The condition (18) is considerably weaker than that complying with the Rayleigh hypothesis. In particular, all convex bodies satisfy the condition (18). The described procedure shows excellent convergence and can be generalized to the problems of wave scattering by several bodies and by gratings too.

Similar approach can be suggested for solving the problems of wave scattering with a periodical surface [10]. Consider for definiteness sake the case of Dirichlet boundary conditions. The diffraction field $u^e(x,y)$ in the region $y > y_e$ can be represented by the expansion (6) characterized by the expression $R^e_m \equiv 2g_0(w_m)/bv_m$, where

$$g_0(w) = \int_{-b/2}^{b/2} Q(x)\exp[iwx + ivf(x)]dx \qquad (19)$$

is the pattern for a single period [6],

$$v = \sqrt{k^2 - w^2},\ \mathrm{Im}\, v \le 0,$$

and

$$\begin{aligned} Q(x) &\equiv Q^0(x) + Q^1(x) = \\ &= \frac{i}{4}\Big[\frac{\partial(u^0 + u^e)}{\partial y} - \\ &\quad - f'(x)\frac{\partial(u^0+u^e)}{\partial x}\Big]\Big|_{y=f(x)}, \end{aligned}$$

$u^0(x,y) = \exp(-ikx\sin\theta + iky\cos\theta)$ is the incident plane wave.

Taking advantage of the expressions (6) and (19), we can obtain the following algebraic system for coefficients $g_m \equiv g_0(w_m)$

$$g_n = g^0_n + \sum_{m=-\infty}^{\infty} f_{mn} g_m, \qquad (20)$$

where

$$g^0_n = \int_{-b/2}^{b/2} Q^0(x)\exp[iw_n x + iv_n f(x)]dx$$

and

$$\begin{aligned} f_{mn} &= \frac{(m-n)pw_n}{4\pi i v_m}\int_{-\pi}^{\pi} f(\frac{t}{p})\exp[i(n-m)t]dt + \\ &\quad + \frac{1}{2}\delta_{mn}, \end{aligned}$$

where $p = 2\pi/b$. We see that the matrix elements of the system (20) are expressed in terms of a single integral, offering significant advantages over the systems resulting when so-called current method is used [13].

Eliminating from (20) the unknown quantity g_0 and passing on to new unknown coefficients with the help of substitution

$$g_n = |n|^{1/2}\exp[(2\lambda/b)|n|\sigma_1]x_n$$

where $\sigma_1 = ky_e/2$, we obtain a new algebraic system

$$x_n = x^0_n + \sum_{m=-\infty}^{\infty}(1-\delta_{m0})F_{mn}x_n, \qquad (21)$$

$n = \pm 1, \pm 2, \ldots$, solvable through the utilization of the reduction method when [10]

$$\sigma_2 > \sigma_1, \qquad (22)$$

where $\sigma_2 = ky_i/2$. In (21),

$$\begin{aligned} x^0_n &= (g^0_n + 2g^0_0 f_{0n})|n|^{-1/2}\times \\ &\quad \times\exp[-(2\lambda/b)|n|\sigma_1], \\ F_{mn} &= (f_{mn} + 2f_{m0}f_{0n})\sqrt{\left|\frac{m}{n}\right|}\times \\ &\quad \times\exp[(2\lambda/b)(|m|-|n|)\sigma_1]. \end{aligned}$$

Fig. 1 exemplifies the realization of the above procedure. The problem of surface wave propagation along a sinusoidal corrugation $f(x) = l\cos(2\pi x/b)$ has been examined. The Figure shows the variation of $h/k - 1$, where h is the cross wavenumber, with the quantity of terms of the field expansion. The graphs are constructed for various values of parameter l characterizing the depth of the corrugation. As can be seen in Fig. 1, the computing algorithm becomes unstable when the condition (22) is violated (for $l/b > 0.105\ldots$): increasing the number of terms of expansion causes the divergence of the result.

The condition (22) is significantly less rigorous than that complying with the Rayleigh hypothesis. For example, for a cycloid described by equations

$$\begin{aligned} &x = a(t + \tau\cos t);\ y = a\tau\sin t, \\ &a > 0,\ 0 \le \tau \le 1;\ 0 \le t \le 2\pi. \end{aligned}$$

the Rayleigh hypothesis is satisfied only when $\tau < 0.2784613\ldots$, whereas the condition (22) holds true for $0 < \tau \leq 1$ [10].

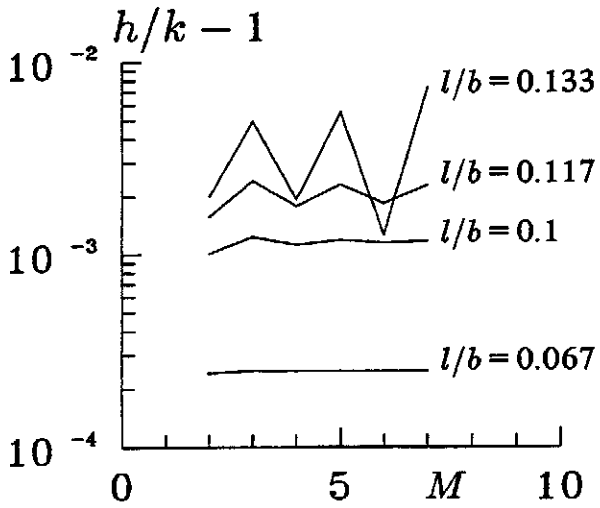

Figure 1. Variation of $h/k - 1$ with the parameter M

3. Auxiliary current method

The auxiliary current method using the *a-priory* information on analytical solution properties works very well [14]. This method is applicable for solving a great variety of diffraction and wave scattering problems [15–17]. Let us take up this method, using as an example the problem of electromagnetic wave diffraction by a perfectly conducting body placed in a uniform isotropic medium. In this case, the boundary-value problem can be reduced to the solving of the integral equations of the form

$$\frac{i}{4}\int\limits_{\Sigma} J(\vec{r}_\Sigma) H_0^{(2)}(k|\vec{r}-\vec{r}_\Sigma|)d\sigma = -u^0(\vec{r}), \qquad \vec{r} \in S \tag{23}$$

for two-dimensional, and

$$\frac{i\sqrt{\mu_a\varepsilon_a}}{4\pi k}\left[\nabla\times\left(\nabla\times\iint\limits_{\Sigma}\vec{J}(\vec{r}_\Sigma)\times \times\frac{\exp(-ik|\vec{r}-\vec{r}_\Sigma|)}{|\vec{r}-\vec{r}_\Sigma|}d\sigma\right)\right]_\tau = \vec{E}^0_\tau(\vec{r}) \tag{24}$$

for three-dimentional cases, where μ_a, ε_a are the magnetic and dielectric constants of the medium; $\vec{A}_\tau$ is the component of the vector $\vec{A}$, tangential to S. It is not difficult to prove the statement as follows [14,15]: *let us assume that Σ is an arbitrary closed nonresonant Liapunov surface inside D_i. Then the necessary and sufficient condition on solvability of the equations of the form (23), (24) in $L_p(\Sigma)$, $p \geq 1$, implies that the surface Σ should enclose the $\overline{\mathcal{E}}_0$ of singularities of scattered field analytical continuation into D_i.*

In one way or another Eqs. (23), (24) can be transformed into algebraic equations solvable with a computer. The method of algebraization employed most often is the substitution of the integrals in the left-hand sides of the equation for sums with the rectangular formula, followed by equating the left and right members at the collocation points. As a result, we obtain highly efficient and fast algorithms known as "discrete source method". At this point, it is worth noting that the following theorem is valid [18]: *the sequence of discrete source amplitudes is limited in l_p $(p > 1)$ if and only if the discrete source support (surface Σ) contains the set $\overline{\mathcal{E}}_0$ of singularities of scattered field analytical continuation into D_i.*

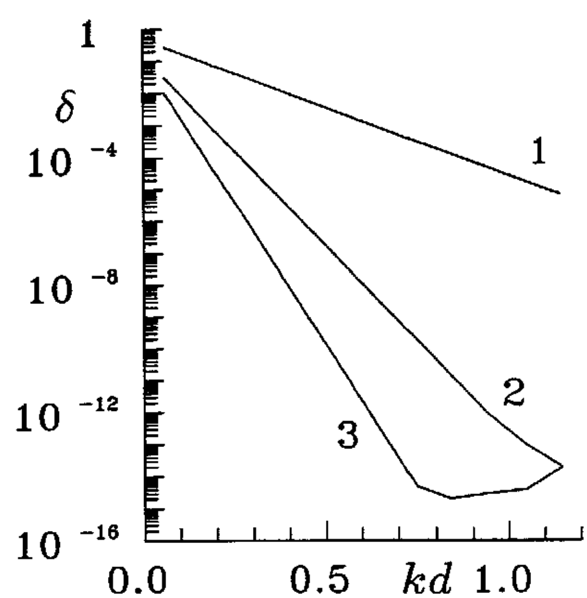

Figure 2. Variation of discrepancy δ with the parameter kd

We will now examine an example to illustrate the importance of meeting the conditions imposed by the above theorems. Figs. 2 and 3 show the results of solving the problem of plane wave scattering by an elliptic cylinder with half-axes measuring $ka = 3$, $kb = 1.2$, $kf = 2.7495\ldots$ ($2f$ — interfocal distance) using the discrete source method. In this problem, the set $\overline{\mathcal{E}}_0$ is the interfocal distance [7]. Fig. 2 shows the dependence of boundary condition discrepancy δ on the parameter kd characterizing the spacing between the auxiliary contour Σ and the cylinder cross-section contour S, a co-focal ellipse with semiminor axis $b_1 = b - d$ being chosen for Σ. Numbers 1,2,3 designate graphs plotted for 30, 80 and 120 sources, respectively. It can be seen that the auxiliary contour, when properly chosen (so that it would encompass the interfocal distance), makes it possible to carry out computations with high precision with no algorithm destruction, even when the algebraic system is rather large. The value of boundary condition discrepancy is the most accurate and highly sensitive indicator of the validity of obtained solution, since integral characteristics (e.g. scattering pattern) are predicted with considerably higher degree of accuracy than that afforded by boundary conditions. However, the norm of current on the auxiliary contour also makes it possible to estimate the algorithm stability.

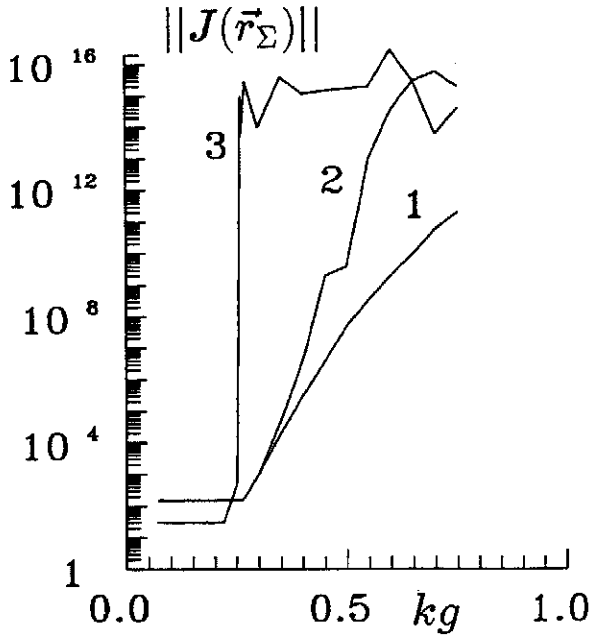

Figure 3. Dependence of norm of current $||J(\vec{r}_\Sigma)||$ on parameter kg

Fig. 3 shows the dependence of the norm of current $||J(\vec{r}_\Sigma)||$ on the parameter kg, determined with the equation $g = a - a_1$, where a_1 is the semimajor axis of the auxiliary elliptic contour, whose semiminor axis is taken to be $kb_1 = 0.7$. The curves 2 and 3 correspond to the algebraic system dimensions 60×60 and 120×120. The curve 1 corresponds to the case when the number of collocation points is five times greater than the number of discrete auxiliary sources (the algebraic system dimensions are 60×300). It can be seen that once the auxiliary contour no longer encompasses the interfocal distance (when $kg > 0.25$), there comes the computing catastrophe: the norm of current increases exponentially and the more so with the increase of the system dimensions, i.e. when the potential accuracy of the analysis becomes higher (see Fig. 2). This effect is known in antenna syntes theory as a superdirectivity phenomena (see [8]).

The equations of the form (23), (24) can be solved by other means. For example, the auxiliary surface Σ can be divided into regions where the sought-for current $J(\vec{r}_\Sigma)$ is approximated with splines and the equation kernel is integrated. Using such approaches, it is possible to obtain more precise algorithms requiring less computing power. Discussed in [19] is a version of the auxiliary current method using strip currents as auxiliary ones, i.e., in essence, the current $J(\vec{r}_\Sigma)$ is approximated with the piece-constant function.

From the above discussion it follows that application of the auxiliary current method is based on a) the separation of two sets: the set, for which the boundary conditions are established (scatterer surface), and the set representing the carrier of radiating currents (auxiliary surface); b) meeting the requirement for the auxiliary surface to encompass the singularities of the scattered field analytical continuation.

There is a variety of diffraction problems involving the study of wave scattering by bodies with a piecewise analytical boundary. In this case, while the use of the auxiliary surface entirely contained inside the scatterer is not strictly correct, this approach is nevertheless feasible to obtain the quasi-solution. Let us determine the error responsible for meeting the boundary condition, when obtaining the quasi-solution of the two-dimensional problem on scattering of the plane monochromatic wave

$$u^0(x, y) = \exp(-ik(x \sin\varphi - y\cos\varphi))$$

by a perfectly conducted cylindrical body, whose cross-section represents a closed contour composed of arcs and straight-line segments. Here, $\pi/2-\varphi$ is the angle between the direction of the wave propagation and x-axis. Let L_x, L_y be the rectangular overall dimensions; ρ_1, ρ_2, ρ_3, ρ_4 the upper right, upper left, lower left and lower right rounding radii, respectively. The electric field vector is supposed to be perpendicular to the incidence plane and parallel to the axis of the cylinder.

The auxiliary contour Σ is chosen as lying equidistantly with main contour, entirely inside the scatterer at a distance of $\varepsilon < \min(\rho_1, \rho_2, \rho_3, \rho_4)$. In other words, the auxiliary contour encompasses the "main" singularities (center of the rounding arcs) of the scattered field.

Breaking the auxiliary and main contours up into eight characteristic regions and algebraizing Eq. (23) in the simplest manner described above, we reduce the integral equation to a linear algebraic equation system with the square matrix.

Without the loss in generality, we will now consider the case when $\varphi = \pi/2$, $kL_x = 14$, $kL_y = 4$, $k\rho_1 = k\rho_2 = k\rho_3 = k\rho_4 = 1.5$.

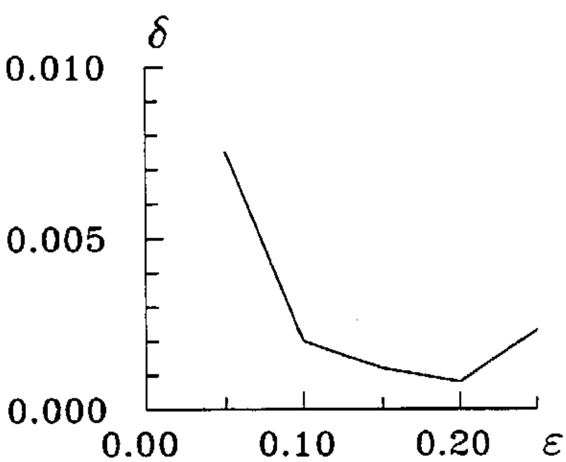

Figure 4. Dependence of the optimal discrepancy δ on ε

Fig. 4 shows the dependence of the optimal residual δ on ε. It can be seen that with $\varepsilon = 0.2$ and the algebraic equation system 155×155, the value of $\delta \sim 10^{-3}$

Making use of the geometry of scatterer, we now examine the case when the insident wave is given in the form

$$u^0(x,y) = \frac{1}{4i} H_0^{(2)}(k|\vec{r} - \vec{r}_0|),$$

i.e. the case of the diffraction of the linear electric field by a body with the piecewise analytical surface; in this context in the form closely approximating the semicircle with its center at $(-x_0, 0)$: $L_x = L_y/2 = y_0$; $\rho_1 = \rho_4 = 0.01y_0$; $\rho_2 = \rho_3 = 0.98y_0$; $ky_0 = -kx_0 = 5.027$. In addition to encompassing the main singularities, the auxiliary contour must encompass the image of the source whose location is determined by [7, 8]. Then, the optimal value of $\delta \sim 10^{-2}$, and the scattering pattern is the same as that reported in [20]. In the examples considered, the surface-carrier of auxiliary currents encompasses not the whole of the diffraction field singularities but only the "strongest" ones, and, therefore, the condition for the above theorems is not compiled with. At the same time, the consequences of such a failure at first glance appear not to be catastrophic. However, this is not the case. For one thing, in the analysis above, the discrepancy of the boundary condition is rather perceptible ($10^{-2} \sim 10^{-3}$). For another, even such a discrepancy is achieved at the cost of significant expenditure of "energy" (here, the quantity $||J(\vec{r}_\Sigma)||$ is of the order $10^5 \sim 10^6$). However, taking into consideration the fact that the integral characteristics such as the scattering pattern are estimated at least one order more precisely than the accuracy of the boundary condition, the above analysis provides reason enough to draw a conclusion that the auxiliary current method is also applicable (with not-too-exacting requirements for the accuracy of the analysis) to the problems of wave scattering by bodies with a piecewise smooth surface on the condition that the auxiliary surface encompasses the main field singularities.

4. Adaptive collocation method. Diffraction by a periodic surface

Widespread use of various types of periodic structures is responsible for the demands for development of simple and effective methods to analyze their diffraction characteristics. The most simple and physically adequate is the method based on the scattered field expansion in terms of outgoing plane waves (metaharmonic functions), i.e. in series of the form (6). An expansion of this kind as well as a method for coefficient computing have been reported by Rayleigh in his classical work [21]. Later, the Barantsev-MMM method has been suggested [22, 23], and it is worth nothing that, as it is known in [13], both Rayleigh and Barantsev-MMM methods lead to the same systems of linear equations for expansion coefficients in the case of the normal plane wave incidence on a symmetrical structure. It should be noted, however, that the Rayleigh method involves quadratures to determine the matrix elements with a consequent substantial (excepting some special cases) computational burden.

The collocation approach is a considerably less complex method for determining the scattering coefficients. With no stringent requirements upon computing aids, this method can be effectively used for development of routine software packages. However, as it is in the case of the Rayleigh method, this approach is comparatively simple only for structures whose depth is small when compared with the period. To obtain the convergent algorithm for analysis of deeper structures, it has been suggested [24] to increase significantly (by several fold) the number of collocation points as compared with the number of methaharmonic functions, as well as to minimize the rms discrepancy of the boundary condition. An alternative, more efficient approach, the adaptive collocation method [25], is discussed later. This method is based on the more judicious allocation of the collocation points rather than on their reproduction. Besides, as it is demonstrated later, the best choice of the collocation points is dictated by the singular points of a scattered field, whose location determines the feasibility of the Raleigh hypothesis.

We will now examine the problem on the plane wave diffraction by a surface including the $z =$ const plane cross-section contour C given by the equation $y = h(x)$, where $h(x)$ is the smooth one-valued function. We will approximate the solution with the sum of metaharmonic functions

$$E_z = E_z^{(0)} + \sum_{n=-M+1}^{M-1} R_n^{(M)} u_n(x,y), \quad (25)$$

$$u_n(x,y) = \exp[-i(w_n x + v_n y)], \quad (26)$$

where

$$E_z^{(0)} = \exp[-ik(x \sin\varphi - y\cos\varphi)]. \quad (27)$$

The boundary condition $E_z = 0$ for $y = h(x)$, established at the discrete points $(x_j,\ y_j = h(x_j))$, results in the linear equation system for coefficients $R_n^{(M)}$:

$$\sum_{n=-M+1}^{M-1} R_n^{(M)} u_n(x_j, y_j) = -E_z^{(0)}(x_j, y_j). \quad (28)$$

Evidently, the field computed from Eq. (25) with coefficients $R_n^{(M)}$ determined from Eq. (28) satisfies the wave equation and the boundary Dirichlet condition at the collocation points (as opposed to the precise solution going to zero on the entire contour C). The completeness of the system of functions $\{u_n\}$ [26] implies that for any positive ε in Eq. (25) there are such number M and such coefficients $R_n^{(M)}$ that the approximate solution differs from the exact one by less than ε. Let us demonstrate (with no claim on mathematical strictness) that choosing the collocation points for Eq. (28) in a prescribed manner, we will obtain the sequence of solutions in the form of Eq. (25), convergent to the exact solution when $M \to \infty$.

Let us introduce the complex variable

$$\eta = \exp[2\pi(ix - y)/b], \quad (29)$$

then the contour C goes into the contour Γ defined by the expression $\eta = \exp[2\pi(ix - h(x))/b]$. It can be shown for with $|n| \gg 1$ $u_n(x,y) \approx \eta^n$, when $\eta \geq 0$; and for $n < 0$ $u_n(x,y) \approx (\eta^*)^{|n|}$. Since the behaviour of the sum (25) under condition $M \to \infty$ is governed by its members having $|n| \gg 1$, it would appear reasonable that the convergence of this sum to the exact solution is attributable to the convergence of the field interpolation by generalized polynomials

$$Q_M(\eta) = \sum_{n=0}^{M-1} (A_n \eta^n - A_{-n}(\eta^*)^n) \quad (30)$$

We represent the powers η and η^* involved in Eq. (29) in the form

$$Q_M(\eta) = \sum_{n=0}^{M-1} [B_n \Phi_n(\eta) + B_{-n}\Phi_n^*(\eta)], \quad (31)$$

where $\Phi_n(\eta)$ are the Faber polynomials in the region bounded by the contour Γ, and then replace the polynomials $\Phi_n(\eta)$ with their asymptotic expression corresponding to $n \gg 1$; then

$$Q_M(\eta) \approx \sum_{n=0}^{M-1} (B_n \tau^n + B_{-n}(\tau^*)^n), \quad (32)$$

where $\tau = \Psi(\eta)$, Ψ is the function mapping conformably the exterior of the contour Γ onto the exterior of the unity circle $|\tau| = 1$. We can see that the convergence of the generalized interpolation polynomial $Q_M(\eta)$ is defined by the convergence of the trigonometric interpolation on the segment $[-\pi, \pi]$. As it is known, this process converges with equidistant nodes, then the interpolation by the polynomials (30) for the inverse mapping $\eta = \Psi^{-1}(\tau)$ will provide the convergence at all intermediate points. Thus, the interpolation by the functions $u_n(x, y)$ converges if the collocation nodes (x_j, y_j) are located so that the points $\tau_j = \Psi(\eta(x_j, y_j))$ are uniformly distributed (when $M \to \infty$) along the circle $|\tau| = 1$. The adaptive node distribution takes place with the inverse mapping of the points uniformly distributed along the unity circle onto the plane τ; for determining the collocation points, we use (see [27]) the asymptotical properties of the polynomials $P_N(\eta)$ orthogonal on Γ (Segõ polynomials [28]). For $N \gg 1$

$$P_N(\eta) \approx \sqrt{\Psi'(\eta)}[\Psi(\eta)]^N. \qquad (33)$$

As it is evident from Eq. (33), at the points $\eta_j^{(N)}$ representing the images of the points

$$\tau_p^{(N)} = \exp[i(\pi(2p-1)/2N)],\ p = \pm 1, \ldots, \pm N, \quad (34)$$

with inverse mapping $\eta = \Psi^{-1}(\tau)$, the real part of the function $P_N(\eta)$ goes to zero for $N \gg 1$. In a similar manner, it can be established that the images of zeros in the imaginary part of the function $P_N(\eta)$ are uniformly distributed along the circle $|\tau| = 1$ for $N \gg 1$, consequently, the zeros of the $\operatorname{Re} P_N(\eta)$ and $\operatorname{Im} P_N(\eta)$ can be regarded as the adaptive collocation nodes. The Segõ polynomials can be constructed using the Gram-Schmidt procedure.

It should be emphasized that the convergence of the interpolation process is closely linked to the location of the singular points of the scattered field analytical continuation with respect to the auxiliary contour discussed above [29]. Referring to the above mentioned similarity to the polynomial interpolation, it is an easy matter to describe the convergence of the interpolation process. Consider a cylinder with cross-section Γ. It is suggested that the electric charge with the surface density proportional to the distribution density of the collocation nodes is placed on this cylinder. In order for the interpolation process to converge, there has to be an equipotential for a given charge system containing the contour Γ but not including the singular point of the scattered field analytical continuation. The speed of the interpolation convergence is determined by the distance between the singular point and the equipotential. It should be noted that the equipotential system corresponding to the collocation node adaptive distribution coincides with the equipotentials of the charged metallic cylinder.

The expansion of the scattered field gives significant computing advantages: the expansion coefficients $R_n^{(M)}$ make it possible to compute the field at any point of space over the structure. These coefficients can be effectively computed with the help of the adaptive collocation method. The resultant approximate solution can, in principle, be arbitrarily near to the precise one. Actual accuracy is limited only by the computing aids performance. With the use of the adaption collocation method, increasing the relative depth of the periodical structure calls for computing operations involving large word sizes if we wish to maintain the desired accuracy.

Compared to other computational approaches, the adaptive collocation method effects a considerable saving in the computer time. The distribution of the adaptive nodes does not depend on the frequency, so the computing of frequency dependencies takes comparatively little time.

The adaptive collocation method is closely allied to the methods used for the problems on the interpolation of functions of complex variable, so many ideas of interpolation theory can be adaptable to problems examined herein. Also, it is worth noting that the relation between collocation points and zeroes of orthogonal polynomials may be used for solving the three-dimensional problems.

5. Conclusion

Using the *a-priory* information on the analytical properties of the solution is a well-established (and necessary) technique for solving inverse problem of the scattering theory [8, 30]. In this case, if the solution is sought for an appropriate (corresponding to the problem) class of functions, the construction algorithms for this solution are highly effective [8]. The extension of the methodology for solving the inverse scattering problems to the direct problems has been rewarding. The algorithms for numerical solving the boundary problems in electrodynamics and acoustics, whose development involves the *a-priory* information on the location of the analytical continuation singularities of the secondary (diffraction) field, are mathematically substantiated, very fast and make it possible to carry out the computations with a high degree of accuracy but no compromise in the algorithm stability. It is worth noting that the methods discussed therein distinguish in versatility. Thus, for example, the diagram equations method involves the limitations on the form of the boundary S, which are stated by the inequality expressions (18) and (22), whereas the auxiliary current method is, in fact, free of these limitations. Furthermore, as it is shown in [16], the auxiliary current method can be generalized practically to any problem in diffraction and wave scattering.

Solving the problems with the help of the adaptive collocation method, we employed the flexible expansions [31]; thus, the limitations on applicability of this methodology become, in general, weaker. Of course, from the aforesaid it does not follow that there is no relation between the collocation algorithms and the Rayleigh hypothesis, but this relation is not so direct as was assumed earlier [32, 33]. It is known, for example, that the Rayleigh hypothesis is valid, when the singular points of the field are well off the scatterer boundary. Evidently, when the Rayleigh hypothesis is obeyed, the convergence of the collocation methods takes place not only at the adaptive nodes, but also applies to a wider set of the node distribution. A similar situation, as it known, exists also in the context of the problems on polynomial interpolation of analytical functions [27].

The results provided illustrate, in essence, the fact that the completeness of the basis employed is not sufficient for the convergence of the solution. Performing actual computations, due attention should be given to the techniques for determining the expansion coefficients. With the collocation method used for characterization of the expansion coefficients, the problem is reduced to the interpolation of the solution on the scatterer contour; the convergence of the computing process depends on the selection of the collocation points. The results related to the polynomial interpolation in the complex plane were used for selecting the interpolation nodes. This allowed us to suggest the techniques for selecting the collocation points providing, as numerical experiments had shown, the convergence of the solution both on the boundary and in the far region. It needs to be stressed that the methods under consideration imply that the information on the singular points of the analytical continuation of scattered field is "embedded" in the distribution of the collocation nodes.

References

[1] P. C. Waterman "New formulation of acoustic scattering". *J. Acoustic Soc. Amer.*, vol. 45, pp. 1417–1499, 1969.

[2] "Acoustic, electromagnetic and elastic wave-scattering focus on T-matrix approach". Ed. Varadan, V.-New York: Pergamon Press, 1980.

[3] C. H. Wilcox "An expansion theorem for electromagnetic fields". *Comm. on Pure and Appl. Math.*, vol. 9, pp. 115–134, 1956.

[4] A. G. Kyurkchan "On the exact value of the radius of convergence of the Wilcox series", *Soviet Physics Doklady*, vol. 36, no 8, pp. 567–569, 1991.

[5] A. G. Kyurkchan "Limits of applicability of the Rayleigh and Sommerfeld representations in three-dimensional wave diffraction problems", *Radio Engineering & Electronic Physics*, vol. 28, no 7, pp. 33–41, 1983.

[6] A. G. Kyurkchan "Analytic continuation of the wave field in the scattering of a plane wave from a periodic surface", *Soviet Physics Doklady*, vol. 32, no 2, pp. 118–121, 1987.

[7] A. G. Kyurkchan, B. Yu. Sternin, V. E. Shatalov, "Singularities of continuation of solution of Maxwell's equations", *Soviet Journal of Communications Technology and Electronics*, vol. 37, no 11, pp. 26–43, 1992.

[8] V. F. Apel'tsin, A. G. Kyurkchan, "Analytical properties of wave fields", M.: MGU, 1990 (in Russian).

[9] A. G. Kyurkchan "A new integral equation in the diffraction theory", *Soviet Physics Doklady*, vol. 37, no 7, pp. 338–340, 1992.

[10] A. G. Kyurkchan, "On a new class of equations in diffraction theory", *Radiotekhnika e elektronika*, vol. 38, no 1, pp. 48–58, 1993. (in Russian)

[11] A. G. Kyurkchan, "On a one method of solving of wave diffraction problems by the compact scatterers", *Doklady AN*, vol. 337, no 6, pp. 721–724, 1994 (in Russian).

[12] A. G. Kyurkchan "Solving equations of mathematical physics by reducing them to ordinary differential equations with operator coefficients", *Soviet Physics Doklady*, vol. 37, no 2, pp. 70–72, 1992.

[13] L. A. Wainstein, A. I. Sukov, "Diffraction by a wave surface: comparison of numerical method", *Radio Endineering & Electronic Physics*, vol. 29, no 8, pp.108–114, 1984.

[14] A. G. Kyurkchan, "Representation of diffraction fields by wave potentials and the method of auxiliary currents in problems of

the diffraction of electromagnetic waves", *Soviet Journal of Communications Technology and Electronics*, vol. 31, no 5, pp. 20–27, 1986.

[15] Yu. A. Eremin, A. G. Sveshnikov, "Method of discrete sources in problems of electromagnetic diffraction", M.: MGU, 1992 (in Russian).

[16] I. M. Polishchuk and A. G. Kyurkchan "Fredholm integral equations of the second kind with smooth kernels for approximate solution of problems of wave scattering by bodies", *Soviet Physics Doklady*, vol. 37, no 11, pp. 549–557, 1992.

[17] A. G. Kyurkchan and A. I. Sukov "Singularities of the scattered field and their role in the method of auxiliary sources", *Soviet Physics Doklady*, vol. 33, no 12, pp. 904–905, 1988.

[18] Yu. A. Eremin, A. G. Sveshnikov, "Method of discrete sources in scattering theory", *Moscow University, Computational Mathematics and Cybernetics*, no 4, pp. 1–10, 1992.

[19] A. Boag, Y. Leviatan and A. Boag, "Analysis of scattering from cylinders with a periodically corrugated periphery using a current-model technique", *IEEE Trans. Antennas Propagat.*, vol. AP-41, no 9, pp. 1265–1272, 1993.

[20] S. Srikanth, P. H. Pathak, C. W. Chuang, "Hybrid UTD-MM Analysis of the Scattering by a Perfectly Conducting Semicircular Cylinder", *IEEE Trans. Antennas Propagat.*, vol. AP-34, no. 10, pp. 1250–1257, 1986.

[21] J. W. Strutt (Baron Rayleigh), "On the Dynamical Theory of Gratings", *Scientific papers.* Cambridge: University press, vol. 5. pp. 388–404, 1912.

[22] R. G. Barantsev, "The separation of variables method in scattering problem by a body of arbitrary shape", *Doklady AN*, vol. 147, no 3, pp. 569–570, 1962 (in Russian).

[23] R. I. Masel, R. P. Merrill, W. H. Miller, "Quantum scattering from a sinusoidal hard wall: Atomic diffraction from solid surfaces", *Phys. Rev. B*, vol. 12, no 12, p. 5545–5551, 1975.

[24] H. Ikuno, K. Yasuura, "Improved Point-Matching Method with Application to Scattering from a Periodic Surface", *IEEE Trans. Antennas Propagat.*, vol. AP-21, no 5, pp. 657–662, 1973.

[25] A. I. Kleev, A. B. Manenkov, "The convergence of point-matching techniques", *IEEE Trans. Antennas Propagat.*, vol. AP-37, no 1, pp. 50–54, 1989.

[26] I. N. Vekua, *New Methods for Solving Elliptic Equation.* Amsterdam, The Netherlands: North Holland, 1967.

[27] J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain.* New York: Amer. Math. Soc., Colloquium, 1960, vol.20.

[28] G. Szegõ, *Orthogonal Polynomials.* New York: Amer. Math. Soc., Colloquium, 1959, vol. 23.

[29] V. F. Apel'tsin, A. G. Kyurkchan, "Rayleigh hypothesis and analytic properties of wave fields", *Soviet Journal of Communications Technology and Electronics*, vol. 30, no 4, pp. 97–11, 1985.

[30] D. Colton, R. Kress, *Integral Equation Methods in Scattering Theory.* New York: J. Wiley & Sons, 1983.

[31] C. Lancos, *Applied Analysis.* Englewood Cliffs, NJ: Prentice-Hall, 1956.

[32] L. Levin, "On the Restricted Validity of Point-Matching Techniques", *IEEE Trans. Microwave Theory Tech.*, vol. MTT-18, no 12, pp. 1041–1047, 1970.

[33] P. Maystre, *Progress in optics.* Amsterdam: North-Holland Ph. Publ., vol. 21, 1984.