A General Mutual Inductance Formula for Parallel Non-coaxial Circular Coils

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Abstract **─** In this paper, we present a complete derivation of a new formula for calculating the mutual inductance between two parallel non-coaxial circular coils by using the magnetic vector potential. Although this problem has been studied by Kim et al, their formula is wrong when the two parallel circulars are non-coaxial. The newly derived formula can not only clarifies the origin of the error introduced in Kim's work, but also can be reduced to the well-known F. W. Grover formula. Different from the Grove's formula that has various cases of the singularities, the new formula enables to simplify all the singular cases into single situation. The singularity in our formula has the clearly geometrical significance, i.e., the point located in one projected circular coil overlapped with the center of another one, which can be easily tackled by the principal value integrals. In order to check the validity of our formula, we have investigated the mutual inductance of the circular coil of the rectangular cross section, and the results demonstrate the correctness of our formula as compared with the other approaches, such as the Neumann's formula, Grover's formula and other published data. In addition, our formula is capable of handling the topics involving the circular coils such as the coils of the rectangular cross section, wall solenoids, disk coils and circular filamentary coils etc. In conclusion, our newly derived formula is of great importance for the electromagnetic applications concerning the calculation of the mutual inductance of the two parallel non-coaxial circular coils.

Index Terms **─** Filament method, Grover's formula, Kim's formula, magnetic vector potential, mutual inductance, Neumann's formula, non-coaxial coils.

I. INTRODUCTION

Circular coils are widely used in various electromagnetic applications such as metal detector [1], inductive coupled contactless energy transfer (CET) system [2], radio frequency identification (RFID) [3], biomedical engineering [4] and the magnetic force etc. [5-7]. The mutual inductance between the primary and secondary coils is an important parameter which determinates the current and electromotive force induced in the secondary windings [8], when the machines are of the induction type. If the machine is such that its secondary is not supported by any means, the axis of the secondary windings may not be the same as that of the primary coils. The mutual inductance between the coaxial circular filaments has been quite thoroughly calculated by a number of authors since the time of Maxwell, and the accuracy of these calculations meets the requirements in practice [9-11]. Many works were based on the Neumann's formula and Biot-Savart law [12-14]. Formula for circular loops with parallel axes has been derived by Butterworth [15] and Snow [16]. Unfortunately, these formulae were slowly convergent and not usable with a wide range of parameters. Using Butterworth's formula, Grover developed a general method to calculate the mutual inductance between circular coils located at any position with respect to each other [10, 11].

Nowadays, with the availability of the powerful numerical methods, such as the boundary element method (BEM) and the finite element method (FEM), it is convenient to obtain the mutual inductance for almost practical electromagnetic applications. However, it is still an interesting topic to discuss this problem by using analytical and semi-analytical methods in many circumstances, as it considerably simplifies the mathematical procedures, and often leads to a significant reduction of the simulation effort.

In this paper, we derived a new formula to calculate the mutual inductance between two non-coaxial circular coils with parallel axes by using the magnetic vector potential, which has been studied for the first time in [17]. First, our formula is not the same as those obtained by Kim et al. Second, the equation (12) in Kim's work is not correct, as pointed out in reference [18], but the reason for that mistake has never been discussed. Third, our formula reduces to the well-known F. W. Grover formula which is expressed in terms of the general Neumann integral, instead of the magnetic vector potential approach. Two numerical experiments are provided in Section 5 to show the validity of our derivations and arguments.

Submitted On: October 10, 2018
Accepted On: April 6, 2019

II. REVIEW OF BASIC EXPRESSION

Let's consider two circular coils as shown in Fig. 1. The center of the primary coil with radius R_p corresponds to the origin O(0,0,0) of plane XOY, and the circle's axis corresponds to the Z axis. The secondary coil with radius R_S is located in $X'O'Y'$ plane, and the Z and Z' axes are parallel with each other. The distance between the planes of the both coils is c, and the distance between the two corresponding axes is d. The mutual inductance between two closed circular loop l_1 and l_2 is given by the double integral Neumann formula:

$$M_{21} = \frac{\mu_0}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{R_p R_s \cos(\varphi_1-\varphi_2)d\varphi_1 d\varphi_2}{R}, \quad (1)$$

here R is the distance between point P and Q, which is given by:

$$R = [R_p^2 + R_s^2 + c^2 + d^2 - 2R_p R_s \cos(\varphi_1-\varphi_2) - 2R_p d\cos\varphi_1 + 2R_s d\cos\varphi_2]^{\frac{1}{2}},$$

and $\mu_0 = 4\pi\times10^{-7} H/m$ is the magnetic permeability of free space.

In Ref. [11], Grover presented the formula for the mutual inductance between the two coils with parallel axes in terms of the Neumann integral, and in Ref. [18], Babic derived the Grover's formula by using the approach of the magnetic vector potential. The Grover's formula for the mutual inductance is given as follows:

$$M_{21} = \frac{2\mu_0\sqrt{R_s R_p}}{\pi}\int_0^{\pi}\frac{1-\frac{d}{R_s}\cos\varphi}{k\sqrt{V^3}}\Psi(k)d\varphi, \quad (2)$$

where

$$a = \frac{R_s}{R_p}, \quad b = \frac{c}{R_p}, \quad V = \sqrt{1+\frac{d^2}{R_s^2} - 2\frac{d}{R_s}\cos\varphi},$$

$$k^2 = \frac{4aV}{(1+aV)^2 + b^2}, \quad \Psi(k) = (1-\frac{k^2}{2})F(k) - E(k).$$

In the above equations, $F(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kinds, respectively [19]:

$$F(k) = \int_0^{\frac{\pi}{2}}\frac{d\theta}{(1-k^2\sin^2\theta)^{1/2}},$$

$$E(k) = \int_0^{\frac{\pi}{2}}(1-k^2\sin^2\theta)^{1/2}d\theta.$$

Based on the approach of the magnetic vector potential, another solution to this problem has been proposed by Kim et al. [17], their expression for the mutual inductance can be expressed as:

$$M_{21} = \frac{\mu_0 R_s}{2\pi}\int_0^{2\pi}\frac{[(R_p+r)^2+c^2]^{1/2}}{r}\Psi(k)d\varphi_2,$$

$$= \frac{2\mu_0 R_s}{\pi}\int_0^{\pi}\frac{1}{k}(\frac{R_p}{r})^{1/2}\Psi(k)d\varphi_2, \quad (3)$$

here r is the distance between the point P and Z axes, which is given by:

$$r = [(d+R_s\cos\varphi_2)^2 + (R_s\sin\varphi_2)^2]^{1/2},$$

and

$$k^2 = \frac{4R_p r}{(R_p+r)^2+c^2}, \quad \Psi(k) = (1-\frac{k^2}{2})F(k) - E(k).$$

As expected, the formula obtained by Kim et al. [17] should be equivalent to Grover's formula because both formulae are derived from the magnetic vector potential. However, in many examples, the numerical results based on equation (2) and (3) are not consistent with each other, which evidently indicate that at least one of them must be incorrect. In order to clarify this, we re-calculate the mutual inductance between two non-coaxial circular coils with parallel axes by using the approach of the magnetic vector potential, as done by Kim et al. The formula obtained in this work is not the same as Kim's work. However, our formula could reduce to the well-known F. W. Grover formula, which strongly supports that our formula is correct. All the details of our derivation and check for the validation of our formula are provided in section 3.

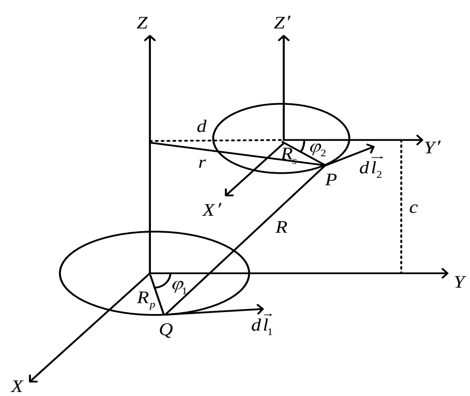

Fig. 1. Configuration of the primary coil and the secondary coil.

III. DERIVATION OF NEW FORMULA

In the following, based on the method of the magnetic vector potential, we will re-calculate the mutual inductance between the two non-coaxial circular coils with parallel axes as shown in Fig. 1. The magnetic vector potential at point P on the secondary coil produced by the primary coil carrying the current I_1 has a tangential component only, and it can be expressed as following:

$$A_\varphi = \frac{\mu_0 I_1}{2\pi}\int_0^{\pi}\frac{R_p\cos\varphi_1}{R}d\varphi_1. \quad (4)$$

Figures 1 and 2 show the dimensions and parameters for calculating the magnetic flux and the mutual inductance between two non-coaxial circular coils. R can be written as follows:

$$R = (R_p^2 + r^2 + c^2 - 2R_p r \cos\varphi_1)^{\frac{1}{2}} . \tag{5}$$

Letting $\varphi_1 = \pi + 2\theta$, so that,

$$\cos\varphi_1 = \cos(\pi + 2\theta) = -\cos 2\theta = 2\sin^2\theta - 1 ,$$

and $d\varphi_1 = 2d\theta$, then,

$$A_\varphi = \frac{\mu_0 I_1 R_p}{\pi} \int_0^{\frac{\pi}{2}} \frac{(2\sin^2\theta - 1)}{[(R_p + r)^2 + c^2 - 4R_p r \sin^2\theta)^{1/2}} d\theta \vec{e}_{\varphi 1} . \tag{6}$$

Substituting the modulus,

$$k^2 = \frac{4R_p r}{(R_p + r)^2 + c^2} ,$$

into equation (6), we can rearrange the integrand to get the following expression for the magnetic vector potential:

$$A_\varphi = \frac{\mu_0 I_1}{\pi k} (\frac{R_p}{r})^{\frac{1}{2}} [(1 - \frac{k^2}{2}) \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 - k^2 \sin^2\theta)^{1/2}}$$

$$- \int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2\theta)^{1/2} d\theta]$$

$$= \frac{\mu_0 I_1}{\pi k} (\frac{R_p}{r})^{\frac{1}{2}} [(1 - \frac{k^2}{2}) F(k) - E(k)] . \tag{7}$$

Using Stokes' theorem, the magnetic flux through the secondary coil due to a current in the primary coil is:

$$\psi_{21} = \int_{s_2} \mathbf{B}_{21} \cdot d\mathbf{s}_2 = \int_{s_2} \nabla \times \mathbf{A}_{21} \cdot d\mathbf{s}_2$$

$$= \oint_{l_2} \mathbf{A}_{21} \cdot d\mathbf{l}_2 , \tag{8}$$

where s_2 and l_2 are the cross section and the perimeter of the secondary circular coil, respectively.

By definition, the mutual inductance between the primary and secondary coils is given by:

$$M_{21} = \frac{\psi_{21}}{I_1} . \tag{9}$$

From (7) - (9), we obtain:

$$M_{21} = \frac{\int_0^{2\pi} A_\varphi \mathbf{e}_{\varphi 1} \cdot \mathbf{e}_{\varphi 2} R_s d\varphi_2}{I_1} =$$

$$\frac{\mu_0 R_s}{\pi} \int_0^{2\pi} \frac{1}{k} (\frac{R_p}{r})^{\frac{1}{2}} [(1 - \frac{k^2}{2}) F(k) - E(k)] \mathbf{e}_{\varphi 1} \cdot \mathbf{e}_{\varphi 2} d\varphi_2$$

$$= \frac{2\mu_0 R_s}{\pi} \int_0^{\pi} \frac{1}{k} (\frac{R_p}{r})^{\frac{1}{2}} \Psi(k) \cos\alpha d\varphi_2 . \tag{10}$$

Obviously, equation (3) and equation (10) are not the same, because $\cos\alpha$ is not equal to 1 in general. Equation (3) is just coincident with equation (10) if and only if two coil axis coincide with each other.

The kernel function of equation (2) is singular in the case $V = 0$. For the case $c \neq 0$ and $d = R_s$, a modified Grover's formula can be adopted, and the Bessel function can be used to solve the singularity case for $c = 0$, $d = R_s = R_p$ and $c = 0$, $d = 2R_s = 2R_p$. However, the kernel function of equation (10) is singular in the case $r = 0$. The principle value integrals of Gaussian numerical integration are adopted for all the singularity case at $r = 0$.

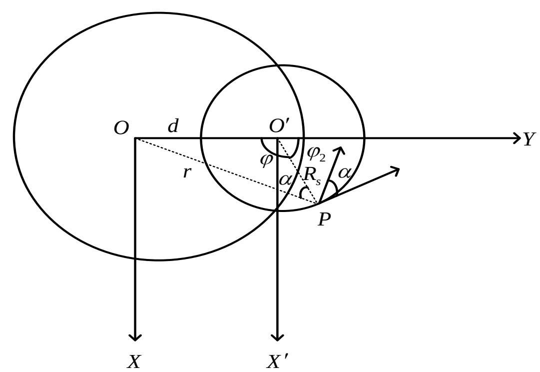

Fig. 2. Top view of the two circular coils.

Let's consider the two circular coils as shown in Fig. 1. The top view of the two circular coils is shown in Fig. 2. Using cosine theorem in triangle $\Delta OO'P$, we obtain:

$$d^2 = r^2 + R_s^2 - 2R_s r \cos\alpha , \tag{11}$$

$$r^2 = d^2 + R_s^2 - 2R_s d \cos\varphi . \tag{12}$$

Substituting equation (11) into equation (10), the mutual inductance between the primary and secondary coils is given by:

$$M_{21} = \frac{2\mu_0 R_s}{\pi} \int_0^{\pi} \frac{1}{k} (\frac{R_p}{r})^{\frac{1}{2}} \psi(k) \frac{r^2 + R_s^2 - d^2}{2R_s r} d\varphi_2 . \tag{13}$$

Substituting equation (12) into equation (13), and using $\varphi_2 = \pi - \varphi$ transforms, the mutual inductance can be re-written as:

$$M_{21} = \frac{2\mu_0 R_s}{\pi} \int_0^{\pi} \frac{1}{k} (\frac{R_p}{r})^{\frac{1}{2}} \psi(k) \frac{r^2 + R_s^2 - d^2}{2R_s r} d\varphi_2 . \tag{14}$$

Introducing additional dimensionless variable:

$$V = \sqrt{1 + \frac{d^2}{R_s^2} - 2\frac{d}{R_s}\cos\varphi} = \sqrt{\frac{R_s^2 + d^2 - 2dR_s \cos\varphi}{R_s^2}}$$

$$= \sqrt{\frac{r^2}{R_s^2}} = \frac{r}{R_s} . \tag{15}$$

Finally, equation (14) indeed reduces to well-known Grover formula (2):

$$M_{21} = \frac{2\mu_0 \sqrt{R_p R_s}}{\pi} \int_0^{\pi} \frac{1 - \frac{d}{R_s}\cos\varphi}{k\sqrt{V^3}} \psi(k) d\varphi , \tag{16}$$

where

$$a = \frac{R_s}{R_p},\ \ b = \frac{c}{R_p},\ \ k^2 = \frac{4aV}{(1+aV)^2 + b^2},$$

$$\Psi(k) = (1 - \frac{k^2}{2})F(k) - E(k),$$

which is the desired proof.

IV. FILAMENT METHOD

In order to verify the accuracy and the computational cost of the expressions for the mutual inductance obtained by using the Neumann formula of equation (1), Grover formula of equation (2), the formula in Kim et al. (also see equation (3)) and our new formula of equation (10), we will present an efficient filament method to calculate the mutual inductance of two circular coils with rectangular cross section.

In order to account for the finite dimensions of the coils, the primary and secondary coils are considered to be subdivided into meshes of filamentary cells as shown in Fig. 3. The cross sectional area of the primary coil is divided into $(2K + 1)$ by $(2N + 1)$ cells, and that the secondary coil into $(2m + 1)$ by $(2n + 1)$ cells. Each cell in each coil contains one filament, and the current density in the coil cross section is assumed to be uniform, so that the filament currents are equal. This means that it is possible to apply equations (1)-(3) and (10) to the pairs of the filament in two coils.

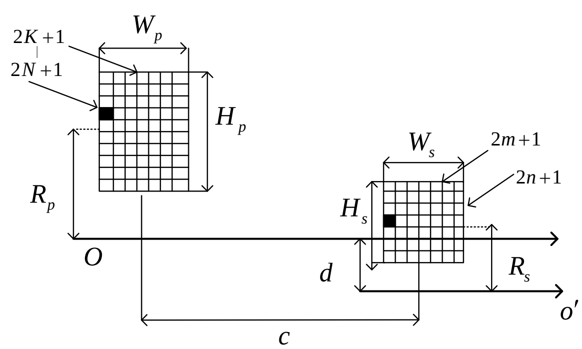

Fig. 3. Configuration of mesh coils.

Using the same procedure given in [17] the mutual inductance can be expressed in the following form:

$$M = \frac{N_p N_s}{(2K+1)(2N+1)(2m+1)(2n+1)} *$$

$$\sum_{g=-K}^{K} \sum_{h=-N}^{N} \sum_{p=-m}^{m} \sum_{l=-n}^{n} M_{ji}(g,h,p,l)\,, \qquad (17)$$

where i and j are the corresponding thin coils of the first and second coil, respectively. M_{ji} is the mutual inductance between i and j filamentary coils. The number of turns in the primary coil is N_p, and that in the secondary coil is N_s:

$$R_p(h) = R_p + \frac{H_p}{(2N+1)}h, \qquad h = -N, \dots, 0, \dots, N\,,$$

$$R_s(l) = R_s + \frac{H_s}{(2n+1)}l, \qquad l = -n, \cdots, 0, \cdots n\,,$$

$$z(g,p) = c - \frac{W_p}{(2K+1)}g + \frac{W_s}{(2m+1)}p\,,$$

$$g = -K, \dots, 0, \dots, K;\ p = -m, \dots, 0, \dots, m\,,$$

$$\Psi(k) = (1 - \frac{k^2}{2})F(k) - E(k)\,.$$

V. NUMERICAL EXAMPLES

Example 1: Two circular coils of rectangular cross section with parallel axes

To validate our formula, we compare the results obtained by Neumann's formula, Grover's formula, the formula in Kim's work and our formula for a given application case. The problem consists of the two circular coils of rectangular cross section with the following dimensions:

(1) Primary coil:

$R_p = 42.5mm$, $W_p = 10mm$, $H_p = 10mm$, $N_p = 150$.

(2) Secondary coil:

$R_s = 20mm$, $W_s = 4mm$, $H_s = 4mm$, $N_s = 50$.

The number of subdivisions was $K = N = 2$, $m = n = 1$.

The mutual inductance between the two circular coils of rectangular cross section with lateral misalignment and parallel axes are calculated and shown in Fig. 4. The mutual inductance as a function of off-center distance d for c=0 is plotted. For the off-center distance d varied from 0.0 to 0.3 m, all the results obtained by Neumann's formula, Grover's formula and our formula are in an excellent agreement. The results by the formula in Kim's work are correct if and only if two coil axis coincidence. However, the results of Kim's formula are wrong when there is a slight misalignment of the two coil's center axes. The value of mutual inductance obtained by equations (1), (2) and (10) all presents a sign reversal (i.e., the sign of the value changes from positive to negative as shown in Fig. 4) when the secondary coil moves far away from the primary coil. There is a negative maximum at d = 0.06 m and the value of the mutual inductance approaches to zero for the large misalignment "d". The characteristics can be explained by the electromagnetic theory; the detailed reason is that the magnetic flux linked by the secondary coil change their orientation outside the primary coil. On the other hand, the mutual inductance obtained by equation (3) slowly approaches zero when the secondary coil is outside the primary coil, and never reach negative values, which is not correct according to electromagnetic theory. Thus, this proves that equation (3) is erroneous for non-coaxial circular coils. Actually, it is exact only when the coils' axes are in coaxial position. In addition, we have compared the calculation time of these approaches, the calculation times of our formula and Grove's formula are 0.37 seconds and 0.32 seconds, respectively, which are comparable and considerably smaller than that of the Neuman's formula (70.08 seconds).

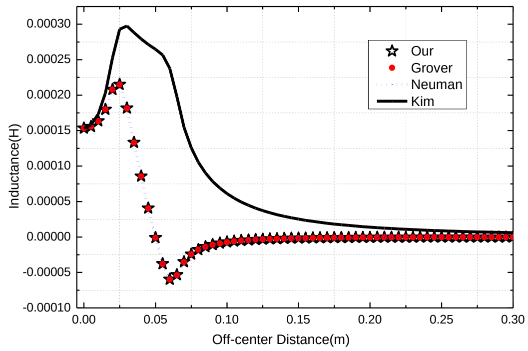

Fig. 4. Mutual inductance as a function of the off-axis displacement d, with axial distance c=0.

Example 2: Two circular coils of rectangular cross section with parallel axes

We further investigate the two reactance coils of rectangular cross section with parallel axes. The coils' dimensions and the number of turns are set as follows: R_p = 7.832cm, R_s = 11.7729cm, W_p = 14.2748cm, W_s = 2.413cm, H_p = 1.397cm, H_s = 4.1529cm, N_p = 1142, N_s = 516, d = 30.988cm, c = 7.366cm, and the number of subdivisions was $K = N = m = n = 12$. Table 1 lists the calculated results by analytical expression and Maxwell software.

From Table 1, one can see that our results agree well with the results obtained by equations (1), (2), the data from experiment and the Maxwell software, which also prove the formula of Kim's work are invalid for non-coaxial circular coils. Importantly, our formula is much easier to handle the singularity as compared to the Grove's formula. In addition, under the same accuracy, our approach only costs 4.524 seconds, which is comparable to Grove's formula (4.712 seconds) and faster than the Neumann's formula (489.047 seconds), Maxwell software (1486 seconds).

Besides, we have also investigated the other examples involving the circular coils such as the coils of rectangular cross section, wall solenoids, disk coils and circular filamentary coils etc. All the results reveal the correctness and effectiveness of our newly derived formula. The calculations of equations (1), (2), (3) and (10) were taken on a personal computer with Intel(R) core (TM) i7-4790 3.6 GHz processor with FORTRAN programming.

Table 1: Mutual inductance of two circular coils of rectangular cross section

Method	Measured [21]	Neumann's	Kim et al.	Grover's	This Work	Maxwell
M (mH)	-1.47	-1.4705	9.1398	-1.4743	-1.4625	-1.4772
Time (s)		489.047	4.377	4.712	4.524	1486

VI. CONCLUSION

In this paper, we re-derive the mutual inductance between two non-coaxial circular coils with parallel axes. The validity of our formula has been verified by comparing with the Neumann's formula, Grover's formula and previously published data. It should be noted that we only need a simple approach to deal with the singularity of the new formula. In singular situation, the Gaussian numerical integration is recommended. In all the cases, our formula is proved to be successful, therefore we can claim that the new formula (10) is the most general formula for the non-coaxial circular coils with parallel axes. The new formula is more intuitive than Grover's formula (2). The new formula can be easily applied to practical engineering applications, which can be served as an alternative tool as compared to the numerical method that requires a lot of memory and CPU time. In conclusion, we have proposed the new formula of the mutual inductance that can be efficiently calculated with only a personal computer, which is particularly useful to large-scale applications.

ACKNOWLEDGMENT

This work was supported in part by Natural Science Foundation of China (No. 61801002), Natural Science Foundation of Anhui Province (No. 1808085QF183) and Key Natural Science Project of Anhui Provincial Education Department No. KJ2018A0015.

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