Methods for the Evaluation of Regular, Weakly Singular and Strongly Singular Surface Reaction Integrals Arising in Method of Moments

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Abstract - The accurate and fast evaluation of surface reaction integrals for Method of Moments computations is presented. Starting at the classification of the integrals into regular and weakly, strongly and nearly singular integrals, appropriate methods are presented that handle each. A Gauss-Legendre quadrature rule evaluates regular integrals. For singular integrals, the singularity is lifted or weakened by an extraction of the singularity, a transform to polar coordinates or a domain transform. The resulting regular integral is in turn solved by a quadrature rule. The different methods are finally applied to an example, and the resulting accuracy tested against the analytical result. The presented methods are general enough to be used as integration methods for integrands with various degrees of singularity and is not limited to Method of Moments.

1. INTRODUCTION
The solution of integro-differential equations by the Method of Moments is directly influenced by the accuracy of the matrix elements corresponding to the reaction integrals. Efficient and accurate methods for the evaluation of the reaction integrals are hence necessary. The numerical methods must fulfill two major requirements. First, the method should converge quickly to ensure a low computation time. Second, the evaluation should be accurate in order to obtain a good overall accuracy.

The evaluation of the reaction integrals gives rise to integrals whose kernel include the free-space Green's function and its derivatives [PM73]. This necessitates the proper treating of four kinds of integrands. First, regular integrals are solved by usual quadrature rules. Second, weakly singular integrands require mathematical transforms before integrating. Third, strongly singular integrands are only accessible to specialized integration methods. Finally, observation points that are located very close to the integrational domain give rise to nearly singular integrals. This paper presents powerful integration methods for each integral.

The first purpose of the paper is to present different methods and their applicability to the different types of integrals. The second purpose is to present a new accurate method to evaluate strongly singular kernels. The third purpose is to carefully compare the appropriate integration methods. To the authors' knowledge, the present complete description of the integration methods is not available in the literature as such.

The organization of the paper is as follows. The following section presents the classification of the kernels. Then, the four methods for regular, nearly singular, weakly singular and strongly singular integrands are presented with an example of a rectangular surface element [HvHW00]. A comparison of the methods closes the presentation.

2. KERNEL CLASSIFICATION
The reaction integrals within a Method of Moments computation involve the integration of the Green's functions or their derivative together with the basis functions and testing functions [PM73].

The free-space Green's function with a source point at $r'$

$$G(\vec{r},\vec{r}') = e^{-j\beta|\vec{r} - \vec{r}'|}/(4\pi|\vec{r} - \vec{r}'|)$$

and its derivative $\nabla G(\vec{r},\vec{r}')$ determine the behavior of the kernels. The distance factor $R$ contained in the Green's function is:

$$R = |\vec{r} - \vec{r}'| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

(1)

and determines if the integrand at the point $P$ is regular (for $\vec{r} \neq \vec{r}'$) or singular (for $\vec{r} = \vec{r}'$).

For the following, the propagation constant $\beta$ is separated into a real and an imaginary part:

$$\beta = \omega \sqrt{\mu_0 (\varepsilon' - j\varepsilon'' - j\kappa/\omega)} = \beta' + j\beta''$$

(2)

with the usual material parameters $\varepsilon$, $\mu$ and $\kappa$ and the angular frequency $\omega$.

One obtains then:

$$G = \frac{e^{-j\beta R}}{4\pi R} = \frac{e^{\beta R}}{4\pi R} \left( \cos(\beta' R) - j \sin(\beta' R) \right)$$

(3)

The derivative $\nabla G(\vec{r},\vec{r}')$ in cartesian coordinates is then:

$$\nabla G = \frac{\vec{r}' - \vec{r}}{R^2} e^{\beta R} \left[ -\cos(\beta' R) + j \sin(\beta' R) \right]$$

$$+ \frac{\vec{r}' - \vec{r}}{R^2} e^{\beta R} \left[ -\beta' \sin(\beta' R) + \beta'' \cos(\beta' R) \right]$$

$$- j(\beta' \cos(\beta' R) + \beta'' \sin(\beta' R))$$

(4)
Consider each term in (3) and (4) when the observation point $P$ moves closer to the source point, or $R \to 0$. In this case, (3) and (4) become singular, the order of their singularity is listed in Table 1. The imaginary part (5) of the Green's function is regular, and further, the derivative of the Green's function has a regular term (6). Mathematically, (5) and (6) can be integrated with no further problems.

The real part of the Green's function (7) and parts (8) of the derivative of the Green's function are weakly singular with terms $\sim 1/R$. This singularity can be easily regularized.

The last term (9) caused by the derivation of the Green's function is strongly singular $\sim 1/R^2$ for $R \to 0$. A regularization must be carried out before an integration can be carried out. For the strongly singular case, the regularisation is more complicated than for the weakly singular case. Appropriate integration methods are necessary.

### 3. Integration Methods

The integration methods for the kernels in Table 1 are chosen according to the following three cases:

- $R > d$: the distance between the observation point $P$ and the source point is greater than a predetermined minimal distance $d$ (regular),
- $R < d$: the distance between the observation point $P$ and the source point is smaller than a predetermined minimal distance $d$ (regular, but nearly singular),
- $R \to 0$: observation point and source point $P$ coincide (weakly and strongly singular).

Table 2 shows a summary of which integration method is applicable for the various source-observation point distances. As seen, special integration methods are needed that correctly treat regular, nearly singular, weakly singular and strongly singular integral kernels. They are in detail:

- **Regular integration**: Solved by a Gauss-Legendre quadrature method. The method is numerically accurate and easy to implement. Section 3.1 sketches the method.
- **Nearly singular integration**: For a distance $R$ smaller than a minimal distance $d$, the integrand changes rapidly and becomes nearly singular. The appropriate integration method is laid out in section 3.4.
- **Weakly singular integration**: The method for weakly singular kernels is outlined in section 3.2. The integration involves either a domain transform, a transform to polar coordinates or an extraction of the singularity.
- **Strongly singular integration**: Two methods in section 3.3 carry out the integration of strongly singular kernels. The methods are composed of the methods for weakly singular and regular kernels. Section 3.3.2 presents a new method that combines the domain transform and the extraction of singularity. Transforms and an extraction of singularity regularize the singularities, the resulting kernels are integrated by the Gauss-Legendre quadrature as in 3.1.
3.1 Regular Integration

A direct Gauss-Legendre quadrature is proposed for the regular parts in Table 2 [Abr70], [Sch93]. In (10), the general Gauss-Legendre quadrature for a general, regular integrand \( F(u_i, u_j) \) is given as:

\[
\iint_{u,v} F(u_i, u_j) \sqrt{\left| g(u_i, u_j) \right|} du_idu_j = \sum_{i,j} \sum_{k,l} F(u_i, u_j) \sqrt{\left| g(u_i, u_j) \right|} w_i w_j.
\]

(10)

\( F \) is a function of two continuous variables \( u_i \) and \( u_j \). They correspond to the surface variables on a surface element in the Method of Moments [HvHW00]. The expression \( \sqrt{\left| g(u_i, u_j) \right|} du_idu_j \) is the differential surface element that expresses the transform of the surface differential into the \( u_i, u_j \) system.

For the numerical integration, the \( u_i, u_j \) are the nodes on the surface in the interval \( u_{i,j} = [-1,1] \). The \( w_i \) and \( w_j \) are the weights for the nodes. Both are listed in the mathematical literature [Abr70] and are not repeated here.

3.2 Weakly Singular Integration

Three different methods are proposed in this section for the numerical computation of weakly singular kernels if \( R \to 0 \):

- section 3.2.1: regularization of the kernel by a transform into polar coordinates
- section 3.2.2: regularization by a domain transform
- section 3.2.3: regularization by an extraction of the singularity and subsequent analytical integration

All three integration methods are presented for the sake of completeness. The methods are later also used for the integration of strongly singular kernels.

As an example consider the weakly singular term \( \frac{(x' - x)^2}{(4\pi R)^2} \beta' \cos(\beta' R) \) (8) on a planar rectangular element as in Fig. 1 with a size of \( \lambda/10 \) which is typical for a surface element in a Method of Moments implementation.

![Figure 1](image1)

**Figure 1** Example geometry to outline integration accuracy.

3.2.1 Transform to Polar Coordinates

The singular point \( P \) is arbitrarily located on the surface element, with its normalized coordinates \( u_i = [-1,1], u_j = [-1,1] \). The weakly singular integrands are regularized by a transform into polar coordinates [GG90]. The process of regularization is depicted in Fig. 2. First, the surface element is partitioned into four rectangles \( V_k \) with the singular point \( P \) located on the corner that is common to all rectangles. Each rectangle is then in turn subdivided into two triangles, the integration is carried out on each of the eight triangles separately.

The transform into polar coordinates \( \rho, \theta \) is

\[
u_i = u_i^p + \rho \cos \theta \]

(11)

\[
u_j = u_j^p + \rho \sin \theta \]

(12)

\[du_idu_j = \rho d\rho d\theta \]

(13)

The integral is then in polar coordinates

\[
\iint_{u,v} F(u_i, u_j) \sqrt{\left| g(u_i, u_j) \right|} du_idu_j = \sum_{k} \iint_{\rho,\theta} F(\rho, \theta(\rho)) \sqrt{\left| g(\rho, \theta(\rho)) \right|} \rho d\rho d\theta
\]

(14)

The weight \( \rho \) that arises due to the transform from the \( u_i, u_j \)-system into the \( \rho, \theta \)-system lifts the singularity and regularizes the integrand. The numerical integration of (14) is carried out by observing that the upper limit \( \rho(\theta) \) is a function of \( \theta \).

For the above example, the regularization of the integrand is shown in Fig. 3. Fig 3a) shows the weakly singular integrand in the \( u_i, u_j \)-system. Fig 3b) to 3e) depict the regularized integrand in polar coordinates. The angle \( \theta \) is defined on the following intervals for the triangles in 3.2.2:

- Fig. 3b): \( 4.02 \leq \theta < 5.18 \) and \( \rho(\theta) = -1.5/\sin(\theta) \),
- Fig. 3c): \( -1.1 \leq \theta < 0.59 \) and \( \rho(\theta) = 0.75/\cos(\theta) \),
- Fig. 3d): \( 0.59 \leq \theta < 2.76 \) and \( \rho(\theta) = 0.5/\sin(\theta) \),
- Fig. 3e): \( 2.76 \leq \theta < 4.02 \) and \( \rho(\theta) = -1.25/\cos(\theta) \).
\[ du, du = 2A^\alpha ds_1 ds_2 = \frac{(t_i + 1)A^\alpha}{4} dt_i dt_z \]  

where \(A^\alpha\) are the surfaces of the triangles in the \(u_i, u_z\)-system. For source and observation points that are not in the centre \(u_i = 0, u_z = 0\) of a surface element, the \(A^\alpha\) are different for each triangle.

A weakly singular integral is hence regularized using the integration in (20):
\[
\int_{-1}^{+1} \int_{-1}^{+1} F(u_i, u_z) \sqrt{g(u_i, u_z)} du_i du_z = \frac{1}{4} \sum_{k=1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} F(t_i, t_z) \sqrt{g(t_i, t_z)} (t_i + 1)A^\alpha dt_i dt_z
\]

The \(\sqrt{g(u_i, u_z)}\) are the weights arising by the transform of the surface differential into the \(u_i, u_z\)-system. The transform of the differentials from the \(u_i, u_z\)-system into \(t_i, t_z\)-system is according to (19) adds the term \((t_i + 1)\) that lifts the singularity and regularizes the integrand.

In Fig. 5, the triangles arising in the regularization are depicted. 5b) to 5e) show the regularized integrands in the \(t_i, t_z\)-system. Again the integrands are very smooth.

### 3.2.3 Singularity Extraction

The singularity of the Green's function can be lifted by subtracting an analytically integrable part. The method extracts the singularity in such a way that the extracted term is analytically integrated as in (21) for a singular point P:
\[
\int_{u_i}^{u_i + 1} du_i du_z = \int_{u_i}^{u_i + 1} \left( \frac{e^{-\rho R}}{R} - \frac{1}{R} \right) du_i du_z + \int_{u_i}^{u_i + 1} \frac{1}{R} du_i du_z
\]

By subtracting the inverse distance, the first integrand in (21) becomes regular. This term is integrated using the usual Gauss-Legendre integration as in section 3.1. The remaining singular integral in \(1/R\) is integrated analytically.
For a linear triangular surface patch, the integration of $1/R$ can be carried out analytically with no approximation. For higher order geometrical surface approximation functions such as the biquadratic [HvHW00], however, the integration cannot be carried out analytically. In this case the integrand is developed into a series. For the $x$-coordinate, one obtains:

$$
(x' - x^p) = \frac{\partial x'}{\partial u_1} \bigg|_{\omega = \omega_1}(u_i - u_i^p) + \frac{\partial x'}{\partial u_2} \bigg|_{\omega = \omega_2}(u_i - u_i^p)
$$

(22)

$y$ and $z$ are analogous. Completing and combining the expressions in $x$, $y$, and $z$ leads to:

$$
\frac{1}{R} = \frac{1}{\sqrt{g_{11}^2 + g_{12}^2 + g_{13}^2}}\left[\ln \left(R + \sqrt{g_{11}(u_i - u_i^p) + \frac{g_{12}}{\sqrt{g_{11}}}(u_i - u_i^p)}\right)
+ \frac{(u_i - u_i^p)}{\sqrt{g_{11}}}\ln \left(R + \sqrt{g_{11}(u_i - u_i^p) + \frac{g_{13}}{\sqrt{g_{11}}}(u_i - u_i^p)}\right)\right]
$$

(23)

The integration according to $u_1$, $u_2$ in (23) is carried out analytically [SC95]. With the Taylor series, one finally obtains:

$$
\int \int \frac{1}{R} du_1 du_2 = \frac{A(\theta)}{\sqrt{A(\theta)^2 + A(\theta)^2 + A(\theta)^2}}
$$

(24)

### 3.3 Strongly Singular Integration

For the integration of strongly singular integrands, the methods of section 3.2 are combined. The combinations yield regular integrands that are dealt with the method described in 3.1.

Two methods lift the singularity:

- The application of a transform to polar coordinates [PG89], [GG90], [GC87] and subsequent extraction of the singularity. This is presented in section 3.3.1.

- A new method in section 3.3.2 combines the domain transform and a subsequent extraction of the singularity. This new method yields very smooth integrands that are accessible to numerical integrations.

As an example, consider the integration of the singular kernel (9) in the geometry in Fig. 1.

#### 3.3.1 Transform to Polar Coordinates and Extraction of Singularity

The transform to polar coordinates, origin at the point $P$, analogous to section 3.2.1, (11) and (12), $u_i = u_i^p + \rho \cos \theta$, and $u_i = u_i^p + \rho \sin \theta$, reduces the order of the singularity in (9) by one by virtue of the integration weight $\rho d\rho d\theta$. The resulting singularity is only weak, and the method for the extraction of the singularity in section 3.2.3 is applicable. The term that determines the singularity in (9) is given by:

$$
\left(x' - x^p\right)/\left(4\pi R^2\right)
$$

(25)

For an analytical integration in $\rho$, the integrand is developed into a series around the point $P$. With (11) and (12), $u_i - u_i^p = \rho \cos \theta$, $u_i - u_i^p = \rho \sin \theta$, one obtains for an $x$-component in (22):

$$
\left(x' - x^p\right) = \rho \left[\frac{\partial x'}{\partial u_1} \cos \theta + \frac{\partial x'}{\partial u_2} \sin \theta\right] \frac{A(\theta)}{\sqrt{A(\theta)^2 + A(\theta)^2 + A(\theta)^2}} + O(\rho^3)
$$

(26)

Analogous to (26), the expressions for the $y$- and $z$-components yield expressions in $A(\theta)$ and $A(\theta)$. With these, one obtains the term that is necessary to express the distance $R$:

$$
A(\theta) = \sqrt{A(\theta)^2 + A(\theta)^2 + A(\theta)^2}
$$

(27)

The distance $R$ depends now on polar coordinates and is given with (26) and (27):

$$
R(\rho, \theta) = \rho^2 A(\theta) + O(\rho^3)
$$

(28)

With (28) and $\left(x' - x^p\right)/R = A(\theta)/A(\theta) + O(\rho)$ one obtains:

$$
\frac{\left(x' - x^p\right)}{R^2} = \frac{A(\theta)}{A(\theta)} + O(\rho) - \frac{1}{\rho^2 A(\theta) + A(\theta)}
$$

(29)

Incorporating the integration weight $\rho$ for the transform of the differentials from the $u_i, u_j$-system into the $\rho, \theta$-system, one obtains the expression:

$$
\frac{1}{4\pi R^2} du_1 du_2 = \frac{1}{4\pi \rho^2} \left(A(\theta) + O(\rho)\right) d\rho d\theta
$$

(30)

The extraction of the singularity in (30) according to (9) produces the integrand:

$$
\frac{1}{4\pi} \frac{A(\theta)}{\rho A^2(\theta)} \frac{\left(x' - x^p\right) e^{\rho^2}}{R^3} + \frac{1}{4\pi \rho A(\theta)}
$$

(31)
As the limit \( O(\rho)/\rho \to O(1) \) in the first parenthesis in (31) exists, the singularity in the first term is lifted and the integrand is regular. The integration is then carried out e.g., by the method in section 3.1 in polar coordinates \( \rho(\vartheta) \) and \( \vartheta \) for all rectangles \( V_I \):

\[
\frac{1}{4\pi} \sum_{I, \rho(\vartheta)} \int_{\vartheta} \left[ \frac{1}{\rho} \frac{A_0(\vartheta)}{\rho A'(\vartheta)} \right] \left( x'(\rho, \vartheta) - x^{(\rho, \vartheta)} \right) e^{\varphi(\vartheta)} \rho \cos(B(\rho, \vartheta)) \right] d\rho d\vartheta
\]

For the evaluation of the extracted term in (31), the Cauchy-limit is computed:

\[
-\frac{1}{4\pi} \lim_{t \to 0} \int_{\rho(\vartheta)} \frac{A_0(\vartheta)}{\rho A'(\vartheta)} \left( \ln \rho(\vartheta) - \frac{\rho(\vartheta)}{A(\vartheta)} \right) d\vartheta
\]

Now, introduce an \( \epsilon \) vicinity using polar coordinates:

\[
R^2 = \rho^2 = (x' - x^2)^2 + (y' - y)^2 + (z' - z^2)^2
\]

\[
= \rho^2(\vartheta) A^2(\vartheta) + O(\rho^3(\vartheta))
\]

This then:

\[
\rho^2(\vartheta) = \frac{\epsilon}{A(\vartheta)} \frac{1}{1 - \frac{O(\rho^2(\vartheta))}{\epsilon^2}} = \frac{\epsilon}{A(\vartheta)} - O(\epsilon^2)
\]

\[
(35) \text{ in (33), one obtains}
\]

\[
-\frac{1}{4\pi} \lim_{t \to 0} \int_{\rho(\vartheta)} A_0(\vartheta) A'(\vartheta) \left( \ln \rho(\vartheta) - \ln \frac{\rho(\vartheta)}{A(\vartheta)} - O(\epsilon^2) \right) d\vartheta
\]

\[
= -\frac{1}{4\pi} \int_{\rho(\vartheta)} A_0(\vartheta) A'(\vartheta) \left( \ln \rho(\vartheta) - \ln \frac{\rho(\vartheta)}{A(\vartheta)} \right) d\vartheta
\]

with (see also [GC87], [GG90], [HRH97], [PG89])

\[
\frac{1}{4\pi} \lim_{t \to 0} \int_{\rho(\vartheta)} A_0(\vartheta) A'(\vartheta) \left( \ln \rho(\vartheta) - \ln \frac{\rho(\vartheta)}{A(\vartheta)} \right) d\vartheta = 0
\]

The expression in brackets vanishes when carrying out the integration and summation before taking the limit. Then the limit does not need to be carried out anymore. The strongly singular part in (9) caused by the derivative of the Green's function, is now readily treated by (31), (32), and (36).

Consider the integrand in (32). In Fig. 6a), the integrand is shown in the \( u_u, u_z \)-system. It is then transformed into the sub-integrands as in Fig. 6b) to e). The sub-integrands are formed by the sub-division process in Fig. 2. The integration domains are:

- Fig. 6b): \( 4.02 \leq \vartheta \leq 5.18 \) and \( \rho(\vartheta) = -1.5/\sin(\vartheta) \)
- Fig. 6c): \( -1.1 \leq \vartheta \leq 0.59 \) and \( \rho(\vartheta) = 0.75/\cos(\vartheta) \)
- Fig. 6d): \( 0.59 \leq \vartheta \leq 2.76 \) and \( \rho(\vartheta) = 0.5/\sin(\vartheta) \)
- Fig. 6e): \( 2.76 \leq \vartheta \leq 4.02 \) and \( \rho(\vartheta) = -1.25/\cos(\vartheta) \)

3.3.2 Domain transform and extraction of singularity

The new method combines the domain transform and the extraction of singularity in a convenient way. Analogous to 3.3.1, the strongly singular part (9) due to the derivative of the Green's function is treated. The order of the singularity is reduced by one by virtue of the integration weight \( du_i du_s = 2A^2 dR_i dR_s = A^2 (t_i + 1)/4 dt_i dt_s \) in (19) (see 3.2.2). This weight is formed by the transform of the differentials from the \( u_u, u_z \)-system into the \( t_i, t_s \)-system. (15) to (18) show the transform for \( u_u \left(t_i,t_s\right) \) and \( u_z \left(t_i,t_s\right) \).

(15) to (18) show the transform for \( u_u \left(t_i,t_s\right) \) and \( u_z \left(t_i,t_s\right) \).

\[
\sum_{k = 1}^{4} \left[ x'_k(t_i,t_s) - x^{(k)} \right] e^{\varphi(\vartheta)} A_k(t_i + 1) \frac{A^2}{4\pi} \]

(36) In (38), the singular point \( P \) is mapped onto the edge \( t_i = -1 \). In contrast to 3.2.3 and 3.3.1, the distance factor is not expressed in the \( u_u, u_z \)-system around the point \( P \), but rather in the \( t_i, t_s \)-system along the edge \( t_i = -1 \). Using a series expansion, the extracted term is analytically integrable according to \( t_i \). For the \( x \)-component (the \( y \)- and \( z \)-components are analogous) one obtains:

\[
\left( \frac{\partial x'_k}{\partial t_i} \right)_{t_i = -1} \left( t_i + 1 \right) + O(t_i + 1)^2
\]

\[
R^2(t_i, t_s) = \left( t_i + 1 \right)^2 f^2(t_i) + O(t_i + 1)^4
\]

\[
f_i(t_i) = \left( \frac{\partial x'_k}{\partial t_i} \right)_{t_i = -1}^2 + \left( \frac{\partial y'_k}{\partial t_i} \right)_{t_i = -1}^2 + \left( \frac{\partial z'_k}{\partial t_i} \right)_{t_i = -1}^2
\]

The term needed for the extraction of the singularity is given by (39) - (41). With (38), they yield:
\[
\frac{1}{16\pi} \sum_{\alpha} A^{\alpha} \left[ \frac{1}{(t_i + 1) f_i'(t_i)} \frac{\partial \chi_i(t_i,t_j)}{\partial t_i} \right]_{t_i = 1} - \frac{\chi_i(t_i,t_j)}{R_i(t_i,t_j)} e^{\theta(x,t)} \cos(\beta R_i(t_i,t_j))(t_i + 1) \right] (42)
\]

The terms in the brackets are regular

\[
\frac{1}{16\pi} \sum_{\alpha} A^{\alpha} \int_{t_i + 1}^{t_i + 1} \int_{t_i + 1}^{t_i + 1} \left[ \frac{1}{(t_i + 1) f_i'(t_i)} \frac{\partial \chi_i(t_i,t_j)}{\partial t_i} \right]_{t_i = 1} (43)
\]

The integration is carried out using the method in 3.1.

For the second term in (42), the Cauchy limit is formed with the substitution \( t = t_i + 1 \):

\[
\frac{1}{16\pi} \sum_{\alpha} A^{\alpha} \int_{t_i + 1}^{t_i + 1} \frac{1}{(t_i + 1) f_i'(t_i)} \frac{\partial \chi_i(t_i,t_j)}{\partial t_i} \right]_{t_i = 1} (44)
\]

For \( t_i \to 0 \), one introduces an \( \varepsilon \)-vicinity as in (40)

\[
R^\varepsilon = \varepsilon = t_i f_i(t_i) + O(t_i),
\]

one then obtains from (44)

\[
-\frac{1}{16\pi} \lim_{\varepsilon \to 0} \sum_{\alpha} A^{\alpha} \int_{t_i + 1}^{t_i + 1} \frac{1}{(t_i + 1) f_i'(t_i)} \frac{\partial \chi_i(t_i,t_j)}{\partial t_i} \right]_{t_i = 1} d\varepsilon dt_i (46)
\]

with (see (37))

\[
\lim_{\varepsilon \to 0} \sum_{\alpha} A^{\alpha} \int_{t_i + 1}^{t_i + 1} \frac{1}{(t_i + 1) f_i'(t_i)} \frac{\partial \chi_i(t_i,t_j)}{\partial t_i} \right]_{t_i = 1} dt_i \ln(\varepsilon) = 0 (47)
\]

The integration method is now visualized by the example of the integrand in (43) as derived from (9). The strongly singular integrand in the \( u_i, u_i \)-system is in Fig. 7a). The integrands after transform and extraction are shown in the \( t_i, t_i \)-system in Fig. 7b) to 7e). The strongly singular part in (9) is transformed into a very smoothly varying function. Hence, the integration is carried out accurately with only a few nodes ensuring a fast computation.

### 3.4 Nearly Singular Integration

The integrands in Method of Moment computations are nearly singular when structures with a small thickness are to be computed, or if fields very close to a surface must be evaluated. In the first case, the problem is noticeable when setting up the matrix, in the second case when computing the fields after the current sources have been computed. Figures 8a) and 8b) show the two cases. In Fig. 8a), the distance between two discrete surface elements is very small compared to the dimensions of the elements. In Fig. 8b), the distance \( d \) of an observation point \( P \) to the surface is smaller than the geometrical dimensions of the discretization elements.

The straight-forward method to evaluate nearly singular integrals is to increase the number of nodes within the numerical quadrature method. In this case, the number of nodes increases very quickly when the distance \( d \) is reduced. A second method is to solve the nearly singular integrals by subdividing the surface elements in smaller elements, and to continue the subdivision process until convergence is achieved. This ensures that the number of nodes is increased in the vicinity of the observation point. The subdivision process is numerically expensive and is inefficient for very small distances \( d \). Furthermore, the accuracy of a Gauss-Legendre quadrature depends on the number of nodes: The integration using 5 nodes of an element that has been
subdivided into two elements allows a correct integration of a polynomial of the 9th order on each subelement. Without the subdivision process, a polynomial of the 19th order could be integrated.

The efficient approach in [Tel87] and [Hay92] is to transform the surface variables $u_1, u_2$ in such a way that the resulting Jacobian lifts the nearly-singularity. This is to say that the Jacobian should approach the distance $d$ at the critical point [Tel87], [Hay92]. The transform also concentrates the nodes around the location of the near-singularity which leads to a better integration.

4. Example Integrals

The previously presented methods for integrations in Method of Moments computations are now applied to an analytical problem. The integrals in Table 3 are derived from the integrals in Table 1 to yield the analytically integrable functionals (48) and (49). The geometry for the verification is a quadratic surface element that is centered at the origin as in Fig. 1. Each side has a length of 2 m at a wavelength of 20 m. Hence, the length of each side is $\lambda/10$. The observation point $P$ is located at $x = 0.25$ m, $y = 0.5$ m and a variable height $z$. The source point is located at $z = 0$.

Table 3 Example integrals for the verification of the integration methods.

<table>
<thead>
<tr>
<th>Kernel Type</th>
<th>Equation</th>
<th>$R \rightarrow 0$</th>
<th>$R &gt; d$</th>
<th>$R &lt; d$</th>
<th>$R \rightarrow 0$</th>
<th>$R &lt; d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>weakly singular kernel</td>
<td>$\frac{1}{4\pi} \int \int_{S'} \mathbf{\hat{r}}' - \mathbf{\hat{r}} P dS'$</td>
<td>$\rightarrow 4.2$</td>
<td>$\rightarrow 4.1$</td>
<td>$\rightarrow 4.3$</td>
<td>$\rightarrow 4.4$</td>
<td></td>
</tr>
</tbody>
</table>

The integral is solved analytically and numerically, the following error evaluates the quality of the method:

$$\text{error in } \% = \left| \frac{\text{analytical} - \text{numerical}}{\text{analytical}} \right| \cdot 100$$

Note that it is only possible to analytically solve the integrations in Table 3 for planar geometries, not for arbitrary ones.
The integration of (48) is itemized in Table 4. Convergence is fast with increasing number of nodes to the correct value. For \( n > 32 \), no improvements are observed, the result is stable with an increasing number of nodes.

<table>
<thead>
<tr>
<th>( n )</th>
<th>result</th>
<th>error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-0.0519166274</td>
<td>10.39</td>
</tr>
<tr>
<td>4</td>
<td>-0.0570911480</td>
<td>14.64 \times 10^{-1}</td>
</tr>
<tr>
<td>6</td>
<td>-0.0578289883</td>
<td>19.05 \times 10^{-2}</td>
</tr>
<tr>
<td>8</td>
<td>-0.0579257050</td>
<td>23.53 \times 10^{-3}</td>
</tr>
<tr>
<td>16</td>
<td>-0.0579393525</td>
<td>45.73 \times 10^{-7}</td>
</tr>
<tr>
<td>32</td>
<td>-0.0579393378</td>
<td>71.31 \times 10^{-10}</td>
</tr>
<tr>
<td>64</td>
<td>-0.0579393378</td>
<td>26.96 \times 10^{-10}</td>
</tr>
</tbody>
</table>

Table 4   Weakly singular integration of (48) with transform into polar coordinates. Analytical result is -0.0579393378.

4.2.2 Domain Transform

The surface element is sub-divided into four triangles that are in turn mapped to quadrangles. Again, the integration involves \( n \times n \)-nodes corresponding to the number of nodes of the previous example. Table 5 shows the results for the method. Again, convergence to the correct value is observed with a stable result beyond \( n = 32 \) nodes.

<table>
<thead>
<tr>
<th>( n )</th>
<th>result</th>
<th>error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-0.0360523044</td>
<td>37.78</td>
</tr>
<tr>
<td>4</td>
<td>-0.0549348239</td>
<td>51.86 \times 10^{-1}</td>
</tr>
<tr>
<td>6</td>
<td>-0.0578618391</td>
<td>13.38 \times 10^{-2}</td>
</tr>
<tr>
<td>8</td>
<td>-0.0579760027</td>
<td>63.28 \times 10^{-3}</td>
</tr>
<tr>
<td>16</td>
<td>-0.0579393184</td>
<td>33.55 \times 10^{-6}</td>
</tr>
<tr>
<td>32</td>
<td>-0.0579393378</td>
<td>22.88 \times 10^{-10}</td>
</tr>
<tr>
<td>64</td>
<td>-0.0579393378</td>
<td>14.28 \times 10^{-10}</td>
</tr>
</tbody>
</table>

Table 5   Weakly singular integration of (48) with domain transform. Analytical result is -0.0579393378.

4.3 Strongly Singular Integration

The strongly singular integration (49) with the observation point \( P(x = 0.25 \text{ m}, \ y = 0.5 \text{ m}, \ z = 0) \) is carried out according to the Cauchy singular integral. The integration domain is subdivided into four sub-domains as in Fig. 11. This permits the evaluation of the accuracy of the surface integral and the line integral. Fig 11a) shows a discretization into only one surface element. Fig. 11b) discretizes the geometry into four elements, the observation point is then on the corner common to each discretization element. For the discretization in Fig. 11a) the surface integral in (49) vanishes (see 3.3), the integration is completely determined by the line integral. In Fig. 11b) both terms contribute to the result.

The transforms in turn are sketched in Fig. 12.

4.3.1 Transform to Polar Coordinates and Extraction of Singularity

[GG90] evaluates the integral for many combinations of observation points and surface elements. These cases have been verified with the present method. The transform is shown in Fig. 12 (middle).

For a discretization into one element (cf. Fig. 11a), the results are summarized in Table 6. A good convergence is observed for an increasing number of nodes. For \( n = 6 \), an error of almost 0.05 \% is achieved. The method remains stable for a larger number of nodes.

The integration according to Fig. 11b) maps the observation point to \( u_1 = -1, \ u_2 = -1 \) in local coordinates (Fig. 12 middle). For each triangle, \( n \times n \) nodes are used, the line integral uses again \( n \) nodes. Table 7 shows the results of the integration. More nodes are needed for the same accuracy compared to a discretization into one element only.
Table 6  Strongly singular integration of (49) after transform to polar coordinates according to Fig. 10 and 11a). Analytical result is \(-0.052741730991244\).

<table>
<thead>
<tr>
<th>n</th>
<th>surface integral</th>
<th>line integral</th>
<th>error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>-0.0471475986</td>
<td>10.61</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>-0.0524191674</td>
<td>61.16 (-10^2)</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>-0.0527116953</td>
<td>56.95 (-10^3)</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>-0.0527389852</td>
<td>52.06 (-10^4)</td>
</tr>
<tr>
<td>16</td>
<td>0</td>
<td>-0.0527417307</td>
<td>49.36 (-10^6)</td>
</tr>
<tr>
<td>32</td>
<td>0</td>
<td>-0.0527417310</td>
<td>52.06 (-10^10)</td>
</tr>
<tr>
<td>64</td>
<td>0</td>
<td>-0.0527417310</td>
<td>52.06 (-10^10)</td>
</tr>
</tbody>
</table>

Table 7  Strongly singular integration of (49) after transform to polar coordinates according to Fig. 10 and 11b. Analytical result is \(-0.052741730991244\).

<table>
<thead>
<tr>
<th>n</th>
<th>surface integral</th>
<th>line integral</th>
<th>error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.2416862365</td>
<td>-0.1793554832</td>
<td>21.82 (+10^1)</td>
</tr>
<tr>
<td>4</td>
<td>0.1785239348</td>
<td>-0.2481406072</td>
<td>32 (+10^2)</td>
</tr>
<tr>
<td>6</td>
<td>0.1849263635</td>
<td>-0.2398825203</td>
<td>41.99 (+10^4)</td>
</tr>
<tr>
<td>8</td>
<td>0.1868837023</td>
<td>-0.2388539817</td>
<td>14.63 (+10^4)</td>
</tr>
<tr>
<td>16</td>
<td>0.1864218113</td>
<td>-0.2391635614</td>
<td>36.29 (+10^6)</td>
</tr>
<tr>
<td>32</td>
<td>0.1864218178</td>
<td>-0.2388539817</td>
<td>14.63 (+10^9)</td>
</tr>
<tr>
<td>64</td>
<td>0.1864218178</td>
<td>-0.2391635488</td>
<td>19.71 (+10^{10})</td>
</tr>
</tbody>
</table>

4.3.2 Domain Transform and Extraction of Singularity

The integration with domain transform and subsequent extraction of the singularity yields the results in Table 8. The integration shows stable results for an increasing number of nodes. An accuracy of better than 0.01 % is reached for 16 nodes.

A discretization according to Fig. 11b) results in a subdivision as shown in Fig. 12 for the element 1. The singularity is located at \(u_x = -1, u_z = -1\) for each triangle. The integration is carried out according to 3.3.2, \(n \times n\) nodes are used. Table 9 shows the results for both surface and line integrals. A stable convergence is achieved, an error better than 0.01 % is obtained for \(n = 16\).

The integrations in 3.3.1 and 3.3.2 yield both results within the same range of accuracy. For the result shown, the method in 3.3.2 (viz. the domain transform and subsequent extraction of singularity) results in smaller errors for only few nodes.

Table 8  Strongly singular integration of (49) by domain transform according to Fig. 11a). Analytical result is \(-0.052741730991244\).

<table>
<thead>
<tr>
<th>n</th>
<th>surface integral</th>
<th>line integral</th>
<th>error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.0883362945</td>
<td>-0.0615749331</td>
<td>33.74 (+10^2)</td>
</tr>
<tr>
<td>4</td>
<td>0.1783554832</td>
<td>-0.2481406072</td>
<td>32 (+10^3)</td>
</tr>
<tr>
<td>6</td>
<td>0.1849263635</td>
<td>-0.2398825203</td>
<td>41.99 (+10^4)</td>
</tr>
<tr>
<td>8</td>
<td>0.1868837023</td>
<td>-0.2388539817</td>
<td>14.63 (+10^4)</td>
</tr>
<tr>
<td>16</td>
<td>0.1864218113</td>
<td>-0.2391635614</td>
<td>36.29 (+10^6)</td>
</tr>
<tr>
<td>32</td>
<td>0.1864218178</td>
<td>-0.2388539817</td>
<td>14.63 (+10^9)</td>
</tr>
<tr>
<td>64</td>
<td>0.1864218178</td>
<td>-0.2391635488</td>
<td>19.71 (+10^{10})</td>
</tr>
</tbody>
</table>

Table 9  Strongly singular integration of (49) by domain transform according to Fig. 12. Analytical result is \(-0.052741730991244\).

<table>
<thead>
<tr>
<th>n</th>
<th>surface integral</th>
<th>line integral</th>
<th>error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.1793554832</td>
<td>-0.2481406072</td>
<td>22.86 (+10^1)</td>
</tr>
<tr>
<td>4</td>
<td>0.2448561289</td>
<td>-0.327592011</td>
<td>77.79 (+10^2)</td>
</tr>
<tr>
<td>6</td>
<td>0.2386698415</td>
<td>-0.327592011</td>
<td>20.31 (+10^3)</td>
</tr>
<tr>
<td>8</td>
<td>0.2391626624</td>
<td>-0.327592011</td>
<td>43.23 (+10^4)</td>
</tr>
<tr>
<td>16</td>
<td>0.2391635488</td>
<td>-0.327592011</td>
<td>19.17 (+10^9)</td>
</tr>
<tr>
<td>32</td>
<td>0.2391635488</td>
<td>-0.327592011</td>
<td>39.71 (+10^{10})</td>
</tr>
</tbody>
</table>

4.4  Nearly singular integrals

As seen in 3.1, the integration using a Gauss-Legendre quadrature when the observation point is close to a surface (see Fig. 1) yields only slowly converging results. A large number of nodes is required. Even for \(n = 32\), the error reaches 3000 % for an observation point located at \(z_P = 0.01\) m. The method in [Tel87], [Hay92] allows a fast integration with a high level of accuracy.

Fig. 13 shows the error obtained with the method in 3.4 for an increasing number of nodes. The shortest distance between an observation and a source point is \(z_P = 0.01\) m. The largest error is for \(n = 16\) at a distance of \(z_P = 0.02\) m and evaluates to 2.1631 %. For \(n = 32\), this error is reduced to 0.2106 %.

Figure 14 shows a comparison between the methods in 3.1 and 3.4 for the same number \(n = 4\) and \(n = 32\). It is clear from this picture that the integration according 3.4 yields much better results for nearly singular integrals.
5. Conclusions

By properly identifying the orders of the singularity with the reaction integrals of the Method of Moments formulation, appropriate integration methods are found to be applicable for each of the integrals. Regular integrals are solved by a Gauss-Legendre quadrature. Weakly singular integrals necessitate a regularization, and are then integrated numerically. Strongly singular integrals are regularized by a two stage process and integrated both analytically and numerically. The correct method ensures for each kernel a fast and stable convergence to the correct values.

The methods are general enough to be used as methods for surface integrals with various degrees of singularity. The methods are not restricted to Method of Moments formulations in Computational Electromagnetics.

BIBLIOGRAPHY


