Diffraction of Obliquely Incident Plane Waves by an Impedance Wedge with Surface Impedances Being Equal to the Intrinsic Impedance of the Medium

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Abstract — Diffraction of plane waves by an impedance wedge with surface impedances equal to the intrinsic impedance of surrounding medium is investigated for oblique incidence case. In the oblique incidence case, the scattering problem cannot be solved explicitly because of the resultant coupled system of functional equations unless the system is decoupled. Therefore under the assumed condition on wedge impedance, these functional equations are decoupled and the expression for the diffraction coefficient is derived as well as the diffracted fields.

Index Terms — Functional equations, impedance wedge, Maliuzhinets theorem, Sommerfeld integrals.

I. INTRODUCTION

In many practical applications, scatterers are partly wedge shaped metallic structures covered by dielectric materials or metallic structures with finite conductivity which can be simulated with impedance boundary conditions. Therefore, the problem of diffraction by an impedance wedge is investigated by a number of scientists and is very important for both civil and military applications.

Diffraction by an impedance wedge was first solved by Maliuzhinets for the normal incidence case [1]. In this solution, the total field was expressed by the integral of an unknown spectral function. The unknown spectral function was determined using the boundary conditions, the edge conditions, and the radiation condition. The fundamental contribution of the Maliuzhinets method is the reduction of the integral equation into a first order functional equation. But for the oblique incidence case, the problem cannot be solved explicitly, since the resultant equations form a coupled functional equations system.

The solutions for the problem under consideration are available only for some limited wedge opening angles and only under some assumption for the surface impedance of this wedge [2-22].

In this study, applying the Leontovich boundary conditions, a coupled differential equations system is derived for the z-components of the fields. Using the similarity transformation, the relevant matrices are diagonalized assuming that the surface impedance is equal to the free space impedance.

The solution for the Helmholtz equation is sought in the form of Sommerfeld integrals. In order to solve the Maliuzhinets functional equations, the Maliuzhinets theorem is applied to the Sommerfeld integrals. Solving the functional equations, the closed form solution is derived and the uniform asymptotic solution is obtained by applying the steepest descent path method to the Sommerfeld integrals. The numerical results are obtained for different wedge opening and incidence angles and they are shown in Figs. 3 through 7.

II. FORMULATION OF THE PROBLEM

The problem under consideration is a wedge with an opening angle of \(2\Phi\), where the edge coincides with the z-axis. The direction of propagation of the incidence wave is specified by the angles \(\beta\) and \(\phi_0\) as shown in Fig. 1. The
incident field is determined by the z-components of the electromagnetic field.

Fig. 1. The geometry of the problem.

Due to the invariance of both the wedge geometry and the impedance with respect to z, the problem can be reduced to a two dimensional problem and the z-components of the electric and magnetic field vectors of the incident wave can be represented as

\[ \tilde{H}_z = H_0 \exp \left\{ -ikr \sin \beta \cos (\varphi - \varphi_0) \right\}, \]  \hspace{1cm} (3)

and

\[ \tilde{E}_z = E_0 \exp \left\{ -ikr \sin \beta \cos (\varphi - \varphi_0) \right\}. \]  \hspace{1cm} (4)

Using the Maxwell’s equations, the field components can be expressed in terms of z-components as follows:

\[ H_r = \frac{1}{iZ_0 k \sin^2 \beta} \left\{ \frac{1}{r} \frac{\partial E_z}{\partial \varphi} - Z_0 \cos \beta \frac{\partial H_z}{\partial r} \right\}, \]  \hspace{1cm} (5)

and

\[ E_r = \frac{i}{Z_0 k \sin^2 \beta} \left\{ \frac{Z_0}{r} \frac{\partial H_z}{\partial \varphi} + \cos \beta \frac{\partial E_z}{\partial r} \right\}. \]  \hspace{1cm} (6)

where \( k = \omega \sqrt{\varepsilon_0 \mu_0} \) is the free space wave number and \( Z_0 \) is the free space impedance given by

\[ Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}. \]  \hspace{1cm} (7)

On the surfaces of the wedge, the Leontovich impedance boundary condition can be represented as

\[ \tilde{E} - (\hat{n} \cdot \tilde{E}) \hat{n} = Z_{1,2} \hat{n} \times \tilde{H}. \]  \hspace{1cm} (8)

Applying boundary conditions result in a matrix equation system defined as

\[ \frac{1}{r} \begin{bmatrix} \frac{\partial H_z}{\partial \varphi} \\ \frac{\partial E_z}{\partial \varphi} \end{bmatrix} = (-1)^{j+1} \frac{ik}{2} A \begin{bmatrix} H_z \\ E_z \end{bmatrix}_{s_j} \]  \hspace{1cm} (9)

where

\[ A = \begin{bmatrix} Z_j/Z_0 & 0 \\ 0 & Z_0/Z_j \end{bmatrix}, \]  \hspace{1cm} (10)

and

\[ B = \begin{bmatrix} 0 & -1 \\ Z_0 & 0 \end{bmatrix}, \]  \hspace{1cm} (11)

and \( j=1,2 \). To obtain the unknown, this coupled matrix system must be diagonalized. Applying similarity transformation to matrix B can produce a diagonal matrix system. To reach this aim the transformation matrix can first be written as

\[ \begin{bmatrix} H_z \\ E_z \end{bmatrix} = P \begin{bmatrix} u \\ v \end{bmatrix}, \]  \hspace{1cm} (12)

where \( P \) is the similarity transform of matrix B, and is defined as

\[ P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}. \]  \hspace{1cm} (13)

After the necessary manipulations \( P \) can be rewritten as

\[ P = \begin{bmatrix} i & 1/Z_0 \\ Z_0 & i \end{bmatrix}. \]  \hspace{1cm} (14)

By using this similarity transform matrix, matrices A and B are diagonalized as follows.
\[ P^{-1}AP = \begin{bmatrix} \frac{1}{2} \left( \frac{Z_j + Z_0}{Z_0} \right) & \frac{i}{2Z_0} \left( \frac{Z_j - Z_0}{Z_j} \right) \\ \frac{1}{2} \left( \frac{Z_j - Z_0}{Z_0} \right) & \frac{1}{2} \left( \frac{Z_j + Z_0}{Z_j} \right) \end{bmatrix}, \] (15)

and
\[ P^{-1}BP = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}. \] (16)

Within equation (15) the diagonalization condition is observed as
\[ \frac{Z_0 - Z_j}{Z_j} = 0, \] (17)

and finally the decoupled matrix system can be written as
\[ \frac{1}{r} \left[ \frac{\partial u}{\partial \varphi} \begin{bmatrix} \hat{u} \\ 0 \end{bmatrix} + \cos \beta \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \frac{\partial u}{\partial r} \right]_{S_j} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (-1)^{j+1} i k \sin^2 \beta \begin{bmatrix} u \\ v \end{bmatrix}_{S_j}. \] (18)

The solutions for field components are sought in the form of Sommerfeld integrals as
\[ (u, v) = \frac{1}{2\pi i} \int_{\gamma} f_j(\alpha + \varphi) e^{-ikr \sin \beta \cos \alpha} d\alpha, \] (19)

where \( \gamma \) is the Sommerfeld double loops shown in Fig. 2, and \( \alpha \) is the complex planes variable. The calculation of the unknown spectral functions represented by \( f_j \) is given in the section titled far field solution.

Application of the Malyuzhinets’ theorem to the functions \( u \) and \( v \) gives the following equations for \( \varphi = \pm \Phi \).

\[ \begin{bmatrix} \sin(\alpha - \theta) - (-1)^j \end{bmatrix} f_1(\alpha \pm \Phi), \] (20)

\[ \begin{bmatrix} \sin(\alpha + \theta) + (-1)^j \end{bmatrix} f_1(-\alpha \pm \Phi) = C_1' \sin \alpha \]

and

\[ \begin{bmatrix} \sin(\alpha + \theta) - (-1)^j \end{bmatrix} f_2(\alpha \pm \Phi), \] (21)

\[ \begin{bmatrix} \sin(\alpha - \theta) + (-1)^j \end{bmatrix} f_2(-\alpha \pm \Phi) = C_2' \sin \alpha. \]

It is known that
\[ \cos \left( \frac{\theta}{2} \right) = \frac{\chi_\Phi(\theta + 2\Phi)}{\chi_\Phi(\theta - 2\Phi)}. \]

The functional equations for \( f_j(\alpha) \) should be supplemented by an additional condition [13] namely \( f_j(\alpha) \frac{1}{\alpha - \phi_0} \) is regular in \( |\text{Re}\alpha| \leq \Phi \).

Then, the solution can be represented in the following form to satisfy the additional condition.
Here a new function is defined as
\[ F_0(\alpha) = f_{j0}(\alpha) \sigma_{\phi0}(\alpha) \xi_j(\alpha). \] (24)

New unknown spectral functions \( \xi_j(\alpha) \) are introduced to facilitate the solution and \( \sigma_{\phi0}(\alpha) \) is defined as
\[ \sigma_{\phi0}(\alpha) = \frac{\mu \cos(\mu \phi_0)}{\sin(\mu \alpha) - \sin(\mu \phi_0)}, \] (26)
where \( \mu \) is equal to \( \mu = \frac{\pi}{2\Phi} \) and \( \xi_j(\alpha) \) has no poles and zeros in the strip \( \text{Re} \alpha \leq \Phi \). The function \( \sigma_{\phi0}(\alpha) \) satisfies the following relation
\[ \sigma_{\phi0}(\alpha \pm \Phi) = \sigma_{\phi0}(-\alpha \pm \Phi). \] (27)

The known functions \( F_0(\alpha) \) satisfy equation (20) and (21) as \( f_{j0}(\alpha) \). Then, \( \xi_j(\alpha) \) obey the simple functional equations where \( \lambda \) is wavelength sentence.
\[ \xi_j(\alpha \pm \Phi) - \xi_j(-\alpha \pm \Phi) = 0. \] (28)

Since the residue of \( f_j \) at \( \alpha = \phi_0 \) must give the incident field, the following can be written
\[ \text{Res} \ F_0(\alpha) \xi_j(\alpha) \bigg|_{\alpha = \phi_0} = 1, \] (29)
where \( \text{Res} \ f(\alpha) \bigg|_{\alpha = \phi_0} \) is used for the residue of a function \( f(\alpha) \) at a point \( \alpha_0 \). It follows that
\[ \xi_j(\alpha) = \frac{1}{f_{j0}(\phi_0)}. \] (30)

So the solutions for the unknown spectral functions are given by
\[ f_j(\alpha) = \frac{\sigma_{\phi0}(\alpha) f_{j0}(\alpha)}{f_{j0}(\phi_0)}. \] (31)

By substituting (31) into (19) and by evaluating the integral asymptotically by the steepest descent method gives
\[ U(r, \varphi) = \frac{e^{-i \frac{\pi}{2} \gamma^2} e^{i \beta \gamma^2} e^{kr \sin \beta \gamma^2}}{2 \pi \sqrt{2 \pi kr \sin \beta}} \left[ f_1(\varphi - \pi) - f_1(\varphi + \pi) \right], \] (32)

and
\[ V(r, \varphi) = \frac{e^{i \frac{\pi}{2} \gamma^2} e^{i \beta \gamma^2} e^{-kr \sin \beta \gamma^2}}{2 \pi \sqrt{2 \pi kr \sin \beta}} \left[ f_2(\varphi - \pi) - f_2(\varphi + \pi) \right]. \] (33)

When inverse transformation is applied to (32) and (33), \( H_z \) and \( E_z \) are obtained as
\[ H_z = \begin{bmatrix} i & \frac{1}{Z_0} \end{bmatrix}, \] (34)
\[ E_z = \begin{bmatrix} v \end{bmatrix}. \] (35)

More specifically
\[ H_z(r, \varphi) = \frac{e^{-i \frac{\pi}{2} \gamma^2} e^{i \beta \gamma^2} e^{-kr \sin \beta \gamma^2}}{2 \pi \sqrt{2 \pi kr \sin \beta}} \times \left\{ i \left[ f_1(\varphi - \pi) - f_1(\varphi + \pi) \right] + \frac{1}{Z_0} \left[ f_2(\varphi - \pi) - f_2(\varphi + \pi) \right] \right\} e^{-kr \sin \beta \gamma^2} \] (36)

where \( D(\varphi) \) is the diffraction coefficient given as
\[ D(\varphi) = \frac{e^{-i \frac{\pi}{2} \gamma^2} e^{i \beta \gamma^2} e^{-kr \sin \beta \gamma^2}}{2 \pi \sqrt{2 \pi kr \sin \beta}} \times \left\{ i \left[ f_1(\varphi - \pi) - f_1(\varphi + \pi) \right] + \frac{1}{Z_0} \left[ f_2(\varphi - \pi) - f_2(\varphi + \pi) \right] \right\}. \] (37)

where \( f_j(\alpha) \) is defined in (24) and the related functions \( f_{10}(\alpha), f_{20}(\alpha), \sigma_{\phi0}(\alpha), \) and \( \xi_j(\alpha) \) are given in (22), (23), (26), and (30), respectively.

**IV. NUMERICAL RESULTS**

In this paper, diffraction of obliquely incident plane electromagnetic waves by impedance being equal to the intrinsic impedance of surrounding medium is considered. This study is the first to investigate this case. Therefore, we reduced the problem to the normal incidence case taking \( \beta = 90^\circ \) to be able to compare our results with the known studies. In Figs. 3 through 5, it is obvious that our results and the results obtained by Ikiz previously and by Büyükaksoy [17, 23] are very similar.
Fig. 3. Comparison of the results (Φ=120°, ϕ₀=30°). (*): results obtained previously by İkiz (>). results obtained by İkiz in this study.

Fig. 4. Comparison of the the results (Φ=165°, ϕ₀=45°) (*): results obtained previously by İkiz (<): results obtained by İkiz in this study.

In Figs. 6 and 7, we represent the diffraction coefficients for different values of incidence and wedge opening angles.

Fig. 6. Diffraction coefficient $10 \log_{10} |D(\varphi)|$ versus observation angle with Φ=120°, ϕ₀=90°, β=30° (*), 45° (o), 60° (△), 75° (□), 90° (<).

Fig. 7. Diffraction coefficient $10 \log_{10} |D(\varphi)|$ versus observation angle with Φ=157,5°, ϕ₀=120°, β=30° (*), 45° (o), 60° (△), 75° (□), 90° (<).

V. CONCLUSION

The wedge surface impedance being equal to the intrinsic impedance of the surrounding medium, not only presents a convenient mathematical problem, but it can also correspond to a practical structure especially when it is assumed that this condition can be satisfied by choosing the appropriate $\varepsilon_r$ and $\mu$ values for any composite material. From a mathematical point of view, this problem should also be considered as a first step for solving a wedge scattering problem.
with any surface impedance, with plane waves at any random incidence angle.

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**REFERENCES**


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