A Split-Step Pade Solution of 3D-PE Method for EM Scattering from PEC Targets

Z. He and R. S. Chen

Department of Communication Engineering
Nanjing University of Science and Technology, Nanjing, 210094, China
eerschen@njust.edu.cn

Abstract — In this paper, the split-step Pade scheme is introduced to solve the three dimensional parabolic equation for EM scattering problems. By implementing the finite differential method, the calculation can be taken from plane to plane along the paraxial direction and a sparse-matrix equation needs to be solved in each transverse plane. In this way, the computational resources can be saved significantly when compared with the rigorous numerical methods. Numerical results demonstrate that the proposed method can obtain accurate results at wider angles up to 45°.

Index Terms — Electromagnetic scattering, parabolic equation method, split-step Pade.

I. INTRODUCTION

The parabolic equation (PE) method has been used as an efficient tool to analyze the EM scattering problems for a few decades [1-5]. The parabolic equation is an approximation of the wave equation and it is traditionally computed with first order Taylor expansion. By using the finite differential method (FD), the parabolic equation can be solved plane by plane along the paraxial direction. In other words, a three dimensional problem can be converted into a series of two dimensional problems to be solved by the standard parabolic equation (SPE) method. Therefore, less computational resources are needed for the SPE method than the rigorous numerical methods [6-8], such as the method of moments (MoM), the finite-difference time-domain (FDTD), and the finite element method (FEM). However, the standard PE method is a narrow-angle approximation which can only get accurate bistatic RCS results at angles within 15° of the paraxial direction.

Several kinds of high-order approximations have been introduced to the parabolic equation for wider angle bistatic computation [9-15]. These high-order approximations are based on higher order Pade approximations of a composition of the exponential or square-root functions. Both the Pade(1,1) and the Claerbout approximations were applied to the parabolic equation, which can obtain accurate results at angles within 25° of the paraxial direction for the analysis of three dimensional EM wave propagation [9]. In [10-11], the Pade(2,1) and Pade(2,2) approximations are accurate at angles even more than 40° of the paraxial direction. However, both the difference accuracy and the computation efficiency will become low with the order of Pade approximation increasing. Therefore, more efficient approximations should be developed. By using the split-step Pade scheme, a high order Pade approximation can be split as a summation of several lower order Pade approximations [12-15]. As a result, both the computational accuracy and efficiency can be guaranteed. The split-step Pade scheme was firstly introduced to solve the Helmholtz equation for propagation within optical fibers by Feit and Fleck [15]. Then more work about the split-step Pade based parabolic equation has been done by Collins and Thomson [12-14]. However, all of these works are focus on two-dimension scalar parabolic equation for EM propagating problems.

In this paper, the split-step Pade scheme is introduced to the three dimensional vector parabolic equations for the analysis of EM scattering problems. Accurate bistatic RCS results can be obtained at wider angles up to 45° of the paraxial direction. The inhomogeneous boundary conditions are added on the surface of the scattering target in each transverse plane. Moreover, the perfect matching layers (PML) are used to truncate the computational region. The rotating PE method is also used to obtain the full bistatic RCS curves.

The remainder of this paper is organized as follows. In Section 2, the theory and the formulations are given. Three numerical experiments are presented in Section 3 to show the accuracy and efficiency of the proposed
method. Section 4 concludes this paper.

II. THEORY AND FORMULATIONS

A. Standard parabolic equation method

Suppose that a PEC object in free space is illuminated by a plane wave. The scattered field components \( E_x', E_y', E_z' \) can be solved with the scalar wave equation:

\[
\begin{align*}
\frac{\partial^2 E_x'}{\partial x^2} + \frac{\partial^2 E_x'}{\partial y^2} + \frac{\partial^2 E_x'}{\partial z^2} + k^2 E_x' &= 0 \\
\frac{\partial^2 E_y'}{\partial x^2} + \frac{\partial^2 E_y'}{\partial y^2} + \frac{\partial^2 E_y'}{\partial z^2} + k^2 E_y' &= 0 \\
\frac{\partial^2 E_z'}{\partial x^2} + \frac{\partial^2 E_z'}{\partial y^2} + \frac{\partial^2 E_z'}{\partial z^2} + k^2 E_z' &= 0
\end{align*}
\]

(1)

where \( k \) is the wave number.

When the paraxial direction of the parabolic equation is chosen as the \( x \) axis, the reduced scattered fields \( u_x', u_y', u_z' \) can be defined as:

\[
\begin{align*}
u_x'(x, y, z) &= e^{-ikx} E_x'(x, y, z) \\
\nu_y'(x, y, z) &= e^{-ikx} E_y'(x, y, z) \\
\nu_z'(x, y, z) &= e^{-ikx} E_z'(x, y, z)
\end{align*}
\]

(2)

Substitute Equation (2) into Equation (1), the following equations are obtained:

\[
\begin{align*}
\frac{\partial^2 u_x'}{\partial x^2} + 2ik \frac{\partial u_x'}{\partial x} + \frac{\partial^2 u_x'}{\partial y^2} + \frac{\partial^2 u_x'}{\partial z^2} &= 0 \\
\frac{\partial^2 u_y'}{\partial x^2} + 2ik \frac{\partial u_y'}{\partial x} + \frac{\partial^2 u_y'}{\partial y^2} + \frac{\partial^2 u_y'}{\partial z^2} &= 0 \\
\frac{\partial^2 u_z'}{\partial x^2} + 2ik \frac{\partial u_z'}{\partial x} + \frac{\partial^2 u_z'}{\partial y^2} + \frac{\partial^2 u_z'}{\partial z^2} &= 0
\end{align*}
\]

(3)

After the factorization, we can get the forward parabolic equations:

\[
\begin{align*}
\frac{\partial u_x'}{\partial x} &= -ik(1 - \sqrt{Q}) u_x' \\
\frac{\partial u_y'}{\partial x} &= -ik(1 - \sqrt{Q}) u_y' \\
\frac{\partial u_z'}{\partial x} &= -ik(1 - \sqrt{Q}) u_z'
\end{align*}
\]

(4)

where the pseudo-differential operator \( Q \) is defined as

\[
Q = \frac{1}{k^2} \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + n^2.
\]

Then the solution of Equation (4) can be written as:

\[
\begin{align*}
u_x'(x + \Delta x, y, z) &= e^{ik\Delta x(\sqrt{Q} - 1)} u_x'(x, y, z) \\
u_y'(x + \Delta x, y, z) &= e^{ik\Delta x(\sqrt{Q} - 1)} u_y'(x, y, z) \\
u_z'(x + \Delta x, y, z) &= e^{ik\Delta x(\sqrt{Q} - 1)} u_z'(x, y, z)
\end{align*}
\]

(5)

As shown in Fig. 1, the unknowns in the \((x+\Delta x)\)th transverse plane can be calculated from those at \(x\)th transverse plane. The calculation starts before the scattering target and ends beyond it. Moreover, the perfectly matched layer (PML) is applied to truncate the computational domain in each transverse plane. Finally, the radar cross section (RCS) results are calculated with the reduced scattered fields in the last transverse plane by near-field field conversion.

![Fig. 1. The calculation process of the PE method.](image)

B. Split-step Pade solution of parabolic equation

We substitute a rational approximation for the exponential operator:

\[
e^{ik\Delta x(\sqrt{Q} - 1)} = 1 + \sum_{i=1}^{N} \frac{a_i Q}{1 + b_i Q}.
\]

(6)

Suppose \( N=2 \) in Equation (6), we can rewrite the parabolic equation as:

\[
\begin{align*}
u_x'(x + \Delta x, y, z) &= u_x'(x, y, z) + a_1 Q u_x'(x, y, z) + \frac{a_2 Q}{1 + b_1 Q} u_x'(x, y, z) \\
u_y'(x + \Delta x, y, z) &= u_y'(x, y, z) + \frac{a_1 Q}{1 + b_1 Q} u_y'(x, y, z) + \frac{a_2 Q}{1 + b_1 Q} u_y'(x, y, z) \\
u_z'(x + \Delta x, y, z) &= u_z'(x, y, z) + \frac{a_1 Q}{1 + b_1 Q} u_z'(x, y, z) + \frac{a_2 Q}{1 + b_1 Q} u_z'(x, y, z)
\end{align*}
\]

(7)

where the coefficients are [11-12]:

\[
\begin{align*}
a_1 &= 0.008664 + 0.1710i \\
b_1 &= 0.4929 - 0.1435i \\
a_2 &= -0.008664 + 0.1431i \\
b_2 &= -0.04016 - 0.1409i
\end{align*}
\]

(8)

Define

\[
\begin{align*}
v_{ix}'(x + \Delta x, y, z) &= \frac{a_1 Q}{1 + b_1 Q} u_x'(x, y, z) \\
v_{iy}'(x + \Delta x, y, z) &= \frac{a_1 Q}{1 + b_1 Q} u_y'(x, y, z) \\
v_{iz}'(x + \Delta x, y, z) &= \frac{a_1 Q}{1 + b_1 Q} u_z'(x, y, z)
\end{align*}
\]

(9)

\[
\begin{align*}
v_{ix,n}'(x + \Delta x, y, z) &= \frac{a_1 Q}{1 + b_1 Q} u_x'(x, y, z) \\
v_{iy,n}'(x + \Delta x, y, z) &= \frac{a_1 Q}{1 + b_1 Q} u_y'(x, y, z) \\
v_{iz,n}'(x + \Delta x, y, z) &= \frac{a_1 Q}{1 + b_1 Q} u_z'(x, y, z)
\end{align*}
\]

(10)

It can be seen that the parabolic equation can be solved by solving \( v_{ix}'(x + \Delta x, y, z) \) and \( v_{iz}'(x + \Delta x, y, z) \)...
parabolic equations can be written as follows:

\[
\left[ 1 + b_l \left( \frac{1}{k^2} \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + 1 \right) \right] v_{i,x}^j (x + \Delta x, y, z) = a_l \left[ \frac{1}{k^2} \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + 1 \right] u_i^j (x, y, z)
\]

\[
\left[ 1 + b_l \left( \frac{1}{k^2} \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + 1 \right) \right] v_{i,y}^j (x + \Delta y, y, z) = a_l \left[ \frac{1}{k^2} \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + 1 \right] u_i^j (x, y, z)
\]

\[
\left[ 1 + b_l \left( \frac{1}{k^2} \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + 1 \right) \right] v_{i,z}^j (x + \Delta z, y, z) = a_l \left[ \frac{1}{k^2} \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + 1 \right] u_i^j (x, y, z)
\]

\[
l = 1, 2
\]

When the FD scheme is used Equation (10), the forward vector parabolic equations can be written as follows:

\[
\begin{align*}
& \frac{b_j}{k^2 \Delta y^2} v_{i,x}^{m+1, j, p, q-1} + \frac{b_j}{k^2 \Delta y^2} v_{i,x}^{m+1, j, p, q} \\
& \quad + \left( 1 - \frac{2b_j}{k^2 \Delta y^2} - \frac{2b_j}{k^2 \Delta y^2} + b_j \right) v_{i,x}^{m+1, j, p, q} \\
& \quad + \frac{b_j}{k^2 \Delta y^2} v_{i,x}^{m+1, j, p, q+1} + \frac{b_j}{k^2 \Delta y^2} v_{i,x}^{m+1, j, p,q+1} \\
& \quad = a_l \left[ \frac{1}{k^2 \Delta y^2} \frac{\partial^2}{\partial y^2} + \frac{1}{k^2 \Delta y^2} \frac{\partial^2}{\partial z^2} + 1 \right] u_i^{m+1, j, p, q} \\
& \quad + \left( -2a_l \left( \frac{1}{k^2 \Delta y^2} \frac{\partial^2}{\partial y^2} + \frac{1}{k^2 \Delta y^2} \frac{\partial^2}{\partial z^2} + 1 \right) u_i^{m+1, j, p, q} \\
& \quad + \left( -2a_l \left( \frac{1}{k^2 \Delta y^2} \frac{\partial^2}{\partial y^2} + \frac{1}{k^2 \Delta y^2} \frac{\partial^2}{\partial z^2} + 1 \right) u_i^{m+1, j, p, q+1} \\
& \quad \xi = x, y, z, l = 1, 2
\end{align*}
\]

where \( u_i^{m+1, j, p, q} \) is the reduced scattered field at the point of \( x_m = m \Delta x, y_p = p \Delta y, z_q = q \Delta z \).

The following coordinate transformation is introduced for PML domain [16]:

\[
\hat{y} = y - i \delta \sigma(\xi) \frac{1}{\Delta y} \frac{\partial}{\partial y} \xi
\]

\[
\hat{z} = z - i \delta \sigma(\xi) \frac{1}{\Delta z} \frac{\partial}{\partial z} \xi
\]

where \( \sigma(\xi) = \frac{3}{2\delta} \times \frac{1}{\eta} \times \log \left( \frac{1}{10} \right) \times \delta \) \( \xi \), \( \delta \) is the thickness of the PML and \( \eta \) is the wave impedance.

Similarly, the parabolic equation in the PML domain can be obtained:

\[
(1 + b_j) v_{i,x}^m (x + \Delta x, y, z) + \frac{b_j}{k^2} e_i^j \frac{\partial}{\partial y} \left( e_i^j v_{i,x}^m (x + \Delta x, y, z) \right)
\]

\[
= a_l e_i^j \left( \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) u_i^j (x, y, z)
\]

\[
\xi = x, y, z, l = 1, 2
\]

where \( e_i^j \) and \( e_j^i \) are the first order partial derivative of \( e_i \) and \( e_j \), respectively.

**C. Boundary conditions**

The above scalar parabolic equations are coupled through inhomogeneous boundary conditions on the surface of the scattering target. For the PEC objects, the
tangential component of the total field equals zero on the surface:
\[
\begin{align*}
\left( n_x u_x^i (p) - n_x u_x^s (p) \right) & = -e^{-ikx} \left( n_x E_x^i (p) - n_x E_x^s (p) \right), \\
\left( n_y u_y^i (p) - n_y u_y^s (p) \right) & = -e^{-iky} \left( n_y E_y^i (p) - n_y E_y^s (p) \right), \\
\left( n_z u_z^i (p) - n_z u_z^s (p) \right) & = -e^{-ikz} \left( n_z E_z^i (p) - n_z E_z^s (p) \right),
\end{align*}
\]
(15)

where \( p \) is a point on the surface of the scatterer and \( (n_x, n_y, n_z) \) is the outer normal to the surface at \( p \).

To ensure the unicity of the solution, the divergence-free condition is added [1]:
\[
\frac{i}{2k} \left( \frac{\partial u_x'}{\partial y} + \frac{\partial u_y'}{\partial z} \right) + ik u_x' + \frac{\partial u_y'}{\partial y} + \frac{\partial u_z'}{\partial z} = 0.
\]
(16)

D. Rotating PE method

As shown in Fig. 2 (a), only a narrow-angle RCS result is obtained by a single PE run. Therefore, the rotating PE method [4] is introduced to obtain the full bistatic RCS at different frequencies for the proposed method. The scattering pattern of any angle can be calculated by decoupling the paraxial direction from the direction of the incidence. As shown in Fig. 2 (b), the paraxial direction is fixed at x-axis while both the incident wave and the scattering target are rotated by a specified angle.

In our work, the scattering target is discretized with triangle grids. Then the cuboid grids which are needed by the FD scheme are obtained by taking advantage of the geometry information of the triangle ones. Only a narrow-angle RCS result is obtained by a single PE run. Therefore, the rotating PE method is introduced to obtain the full bistatic RCS result. After rotation, the triangle grids can be calculated directly by the coordinate transformation and then the cuboid grids should be regenerated.

III. NUMERICAL RESULTS

In this section, a series of examples are presented to demonstrate the accuracy and efficiency of the proposed method. All computations are carried out on Lenovo Intel Q9500 (2.83 GHz) with 8GB RAM.

A. The bi-static RCS for a PEC sphere

Firstly, the EM scattering from a PEC sphere is considered at the frequency of 300 MHz with the radius 4 m. The incident angle is fixed at \( \theta_0 = 90^\circ \), \( \phi_0 = 0^\circ \). The model of the PEC sphere is shown in Fig. 3. The transverse \( (y, z) \) plane of the air box is chosen to be \( 20m \times 20m \). There are totally 80 transverse planes to be calculated with 200\times 200 nodes in each transverse plane. In this simulation, all the range steps are chosen to be 0.1 m. As shown in Fig. 4, the bistatic RCS curves of the PEC sphere are compared between the traditional PE method, the proposed Split-Step Pade PE method and Mie Series. It can be seen that there is a good agreement between the Mie Series and the proposed Split-Step Pade PE method at wider angles than the standard PE method (SPE).

![Model of the PEC sphere](image)

![Bistatic RCS result for the PEC sphere](image)
B. The monostatic RCS for a PEC cone
Secondly, the analysis of bistatic RCS is taken for a PEC cone at the frequency of 300 MHz with upper radius 2 m, down radius 4 m and height 4 m. As shown in Fig. 5, the model of the cone is given. The incident angle is fixed at $\theta_{inc} = 90^\circ$ $\phi_{inc} = 0^\circ$. All the range steps are chosen to be 0.1 m and the transverse $(y, z)$ plane of the air box is chosen to be $20m \times 20m$. As a result, there are 40 transverse planes to be calculated with 200$\times$200 nodes in each transverse plane. As shown in Fig. 6, the bistatic RCS curves of the PEC sphere are compared between the traditional PE method, the proposed Split-Step Pade PE method and software FEKO. There is a good agreement between the FEKO and the proposed Split-Step Pade PE method at angles of $45^\circ$ of the paraxial direction while $15^\circ$ for the standard PE method (SPE).

![Fig. 5. Model of the PEC cone.](image)

![Fig. 6. Bistatic RCS result for the PEC cone.](image)

C. Complete bistatic RCS for a PEC aircraft
At last, we consider the EM scattering from an aircraft at the frequency of 2.5 GHz and its maximum size in x, y and z directions are 10 m, 2.75 m and 8.5 m. The incident angle is fixed at $\theta_{inc} = 90^\circ$ $\phi_{inc} = 0^\circ$. In this simulation, the transverse $(y, z)$ plane of the air box is chosen to be $12m \times 12m$. There are 167 transverse planes to be calculated with the range steps of 0.06 m and 200$\times$200 nodes in each transverse plane. As shown in Fig. 7, the complete bistatic RCS results are compared between the proposed Split-Step Pade PE method and software FEKO. Moreover, as shown in Fig. 8, the detailed figure of the bistatic RCS result between $0^\circ$ and $60^\circ$ is given for better comparison. There is a good agreement between them. It should be noted that 4 rotating PE runs are used to obtain the complete bistatic RCS for the proposed method while 7 for the standard PE method. Therefore, the proposed Split-Step Pade PE method is more efficient than the standard PE method for analyzing the bistatic EM scattering problems.

![Fig. 7. Bistatic RCS result for the PEC aircraft.](image)

![Fig. 8. Bistatic RCS result between $0^\circ$ and $60^\circ$ for the PEC aircraft.](image)

IV. CONCLUSION
In this paper, the split-step Pade scheme is used to the parabolic equation for the analysis of EM scattering from electrically large PEC objects. By taking
advantage of the split-step Padé scheme, a high order Padé approximation can be divided into several one-order Padé approximations. Moreover, they can be calculated separately. In this way, high computational accuracy and efficiency can be achieved by the proposed method. The numerical results demonstrate that the proposed method can obtain accurate bistatic RCS results at angles even more than 45° of the paraxial direction.

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Zi He received the B.Sc. degree in Electronic Information Engineering from the School of Electrical Engineering and Optical Technique, Nanjing University of Science and Technology, Nanjing, China, in 2011. She is currently working towards the Ph.D. degree in Electromagnetic Fields and Microwave Technology at the School of Electrical Engineering and Optical Technique, Nanjing University of Science and Technology. Her research interests include antenna, RF-integrated circuits, and computational electromagnetics.

Rushan Chen (M’01) was born in Jiangsu, China. He received the B.Sc. and M.Sc. degrees from the Department of Radio Engineering, Southeast University, China, in 1987 and 1990, respectively, and the Ph.D. degree from the Department of Electronic Engineering, City University of Hong Kong, in 2001. He joined the Department of Electrical Engineering,
Nanjing University of Science and Technology (NJUST), China, where he became a Teaching Assistant in 1990 and a Lecturer in 1992. Since September 1996, he has been a Visiting Scholar with the Department of Electronic Engineering, City University of Hong Kong, first as Research Associate, then as a Senior Research Associate in July 1997, a Research Fellow in April 1998, and a Senior Research Fellow in 1999. From June to September 1999, he was also a Visiting Scholar at Montreal University, Canada. In September 1999, he was promoted to Full Professor and Associate Director of the Microwave and Communication Research Center in NJUST, and in 2007, he was appointed as the Head of the Department of Communication Engineering, NJUST. He was appointed as the Dean in the School of Communication and Information Engineering, Nanjing Post and Communications University in 2009. And in 2011 he was appointed as Vice Dean of the School of Electrical Engineering and Optical Technique, NJUST. Currently, he is a Principal Investigator of more than 10 national projects. His research interests mainly include computational electromagnetics, microwave integrated circuit and nonlinear theory, smart antenna in communications and radar engineering, microwave material and measurement, RF-integrated circuits, etc. He has authored or co-authored more than 260 papers, including over 180 papers in international journals.

Chen is an expert enjoying the special government allowance, Member of Electronic Science and Technology Group, Fellow of the Chinese Institute of Electronics (CIE), Vice-Presidents of Microwave Society of CIE and IEEE MTT/APS/EMC Nanjing Chapter and an Associate Editor for the International Journal of Electronics. He was also the recipient of the Foundation for China Distinguished Young Investigators presented by the China NSF, a Cheung Kong Scholar of the China Ministry of Education, New Century Billion Talents Award. Besides, he received several Best Paper Awards from the National and International Conferences and Organizations. He serves as the Reviewer for many technical journals, such as the IEEE Transactions on Antennas and Propagation, the IEEE Transactions on Microwave Theory and Techniques, Chinese Physics, etc.