Scattering from an Arbitrarily Incident Plane Wave by a PEMC Elliptic Cylinder Confocally Coated with a Chiral Material

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Abstract — An analytic solution to the problem of scattering of a plane electromagnetic wave by a chirally coated elliptic cylinder defined by PECM boundary condition has been obtained by expanding the different electromagnetic fields in terms of appropriate elliptic wave functions and a set of expansion coefficients. The expansion coefficients associated with the transmitted field inside the coating as well as the scattered field outside the coating are unknown and will be obtained by applying the boundary conditions at various layers. Numerical results have been presented graphically to show the effects of chiral and PEMC materials simultaneously on the bistatic width of scattering from coated elliptic cylinder.

Index Terms — Bistatic, chiral, elliptic cylinder, Mathieu functions, PEMC.

I. INTRODUCTION

Since the introduction of PEMC materials in 2005 [1], there has been a lot of research into scattering from different types of both two- and three-dimensional PEMC objects [2-12]. This has recently led to an interest on research involving coated PEMC objects [13-14]. As described in [1], a PEMC medium is a generalized form of a perfect electric conducting (PEC) and a perfect magnetic conducting (PMC) medium in which certain linear combinations of electromagnetic fields become extinct [15], and is definable by a single real-valued parameter known as the PEMC admittance. A null admittance corresponds to a PMC medium and an admittance of infinity corresponds to a PEC medium, when the field magnitudes are finite [16]. A PEMC material acts as a perfect reflector of electromagnetic waves, but differs from PEC and PMC materials due to the fact that it produces a reflected wave with a cross-polarized field component [17-22].

The elliptic cylinder is a geometry that has been extensively analyzed in the literature due to its ability to produce cylinders of different cross sectional shapes, by changing the axial ratio of the ellipse. Moreover, since the elliptic cylindrical coordinate system is one of the coordinate systems in which the wave equation is separable, solutions to problems involving elliptic cylinders can be obtained in closed form.

In this paper, we present the analysis corresponding to the scattering from a chiral coated PEMC elliptic cylinder of arbitrary axial ratio, when it is excited by either a plane wave of arbitrary polarization and angle of incidence. Such solution is valuable, since it can be used for validating solutions obtained using other methods. The analysis and the software used for obtaining the results have been validated by calculating the normalized scattering widths for a PEMC coated elliptic [22] when it is illuminated by a plane wave. It was shown graphically that these results are in very good agreement with the corresponding results obtained using various values of admittances for coated PEMC elliptic cylinder.

II. FORMULATION

Consider a linearly polarized uniform plane electromagnetic wave arbitrarily incident on an infinitely long PEMC elliptic cylinder confocally coated with a chiral material. The semi-major and semi-minor axis lengths of the uncoated cylinder are denoted by $a_c$ and $b_c$, and those of the coated cylinder are denoted by $a$ and $b$, respectively. The coated cylinder is assumed to be located in free space, with the incident wave making an angle $\varphi$ with the negative x-axis of a Cartesian coordinate system as shown in Fig. 1. It is beneficial to define the $x$ and $y$ coordinates of the Cartesian coordinate system in terms of $u$, $v$, $z$ of an elliptical coordinate system where $x = F \cosh u \cos v$, $y = F \sinh u \sin v$, with $F$ being the semi-focal length of the ellipse. A time dependence of $\exp(j\omega t)$ with $\omega$ being the angular frequency, is assumed throughout the analysis, but suppressed for convenience. The analysis is conducted for an incident uniform plane wave of transverse magnetic (TM) polarization. The analysis corresponding to a plane wave...
of transverse electric (TE) polarization can be obtained from that for a plane wave of TM polarization, using duality.

\[
E_z^{\text{inc}} = \sum_{q, n} A_{q n} R_{q n}^{(c)}(c, \xi) S_{q n}(c, \eta),
\]

where \( R_{q n}^{(c)}(c, \xi) \) and \( S_{q n}(c, \eta) \) are the i-th order of the radial and angular Mathieu functions respectively, where \( q = e, o \) stands for even and odd solution, and

\[
A_{q n} = j^n \frac{8\pi}{N_{q n}(c)} S_{q n}(c, \cos \phi),
\]

in which \( c = kF \) with \( k \) being the wavenumber of the medium outside the cylinder and \( N_{q n}(c) \) is the normalization constant associated with \( S_{q n}(c, \eta) \). Using Maxwell’s equations we can expand the incident electric field component as [23]:

\[
H_{z}^{\text{inc}} = \frac{1}{jk Z h} \sum_{q, n} A_{q n} R_{q n}^{(c)}(c, \xi) S_{q n}(c, \eta),
\]

where \( h = F \sqrt{\cosh^2 u - \cos^2 \nu} \), with \( \xi = \cosh u \), \( \eta = \cos \nu \), \( Z \) is the wave impedance of the medium outside the cylinder, and the prime denoting the differentiation with respect to \( u \).

Since the cylinder comprises of a PEMC material, the scattered field consists of both co- and cross-polar components. These can be expanded as:

\[
E_z^s = \sum_{q, n} B_{q n} R_{q n}^{(c)}(c, \xi) S_{q n}(c, \eta),
\]

\[
E_{z}^s = -\frac{1}{k h} \sum_{q, n} C_{q n} R_{q n}^{(c)}(c, \xi) S_{q n}(c, \eta),
\]

\[
H_z^s = \frac{j}{Z} \sum_{q, n} C_{q n} R_{q n}^{(c)}(c, \xi) S_{q n}(c, \eta),
\]

\[
H_{z}^s = \frac{1}{j k Z h} \sum_{q, n} B_{q n} R_{q n}^{(c)}(c, \xi) S_{q n}(c, \eta),
\]

in which \( C_{q n} \) and \( B_{q n} \) are the unknown co- and cross-scattered field expansion coefficients.

The fields within the chiral coating also have both co- and cross-polar components, comprising of left- and right-handed parts. These can be expanded as:

\[
E_z^c = \sum_{q, n} \left( [D_{q n} R_{q n}^{(c)}(c, \xi)] + P_{q n} R_{q n}^{(c)}(c, \xi) \right) S_{q n}(c, \eta),
\]

\[
E_{z}^c = \sum_{q, n} \left( -\frac{1}{k h} [D_{q n} R_{q n}^{(c)}(c, \xi)] + P_{q n} R_{q n}^{(c)}(c, \xi) \right) S_{q n}(c, \eta),
\]

\[
H_z^c = \frac{j}{Z} \sum_{q, n} \left( [D_{q n} R_{q n}^{(c)}(c, \xi)] + P_{q n} R_{q n}^{(c)}(c, \xi) \right) S_{q n}(c, \eta),
\]

\[
H_{z}^c = \frac{1}{j k Z h} \sum_{q, n} \left( -\frac{1}{k h} [D_{q n} R_{q n}^{(c)}(c, \xi)] + P_{q n} R_{q n}^{(c)}(c, \xi) \right) S_{q n}(c, \eta),
\]

where \( D_{q n}, P_{q n}, Q_{q n} \) are the unknown field expansion coefficients, \( c = k_R F, c = k_R F, \) with the wavenumbers \( k_R \) and \( k_L \) corresponding to the right- and left-handed waves inside the chiral medium given by \( k_{R,L} = \omega \sqrt{\mu \epsilon} \pm \omega \mu \epsilon \), in which \( \epsilon \) is the chirality admittance and \( \epsilon \) is the effective permittivity defined by \( \epsilon = \mu + \epsilon \epsilon \), with \( \epsilon \) and \( \mu \) being the permittivity and permeability of the chiral medium, and \( Z \) is the wave impedance of the medium, given by \( Z = \sqrt{\mu/\epsilon} \).

The boundary conditions at the surface \( \xi = \xi \) of the PEMC elliptic cylinder can be written using the PEMC admittance \( M \) as:

\[
E_z^c = E_z^c + E_z^c,
\]

\[
E_{z}^c = E_z^c + E_z^c,
\]

\[
H_z^c = H_z^c + H_z^c,
\]

\[
H_{z}^c = H_z^c + H_z^c.
\]
\[ H_x' + ME_y = 0, \quad (16) \]
\[ H_y' + ME_x = 0. \quad (17) \]

Substituting for the electric field components in (12)-(13) in terms of their respective expansions, we get:

\[
\sum_{q,n} \left[ \left( D_{q,n} R_{q,n}^{(1)}(c,\xi) + P_{q,n} R_{q,n}^{(2)}(c,\xi) \right) S_{q,n}(c,\eta) \right] + \frac{1}{j\kappa h} \sum_{q,n} \left[ \left( D_{q,n} R_{q,n}^{(1)}(c,\xi) + P_{q,n} R_{q,n}^{(2)}(c,\xi) \right) S_{q,n}(c,\eta) \right] = 0,
\]

For both sides of equations (18)-(23) are multiplied by \( S_{q,n}(c,\eta) \) and integrated over \( \eta \) from -1 to 1, then considering the orthogonality of the angular Mathieu functions, we can write these equations after a rearrangement as:

\[
D_{q,n} R_{q,n}^{(1)}(c,\xi) + P_{q,n} R_{q,n}^{(2)}(c,\xi) = 0 \quad (24)
\]
\[
A_{q,n} R_{q,n}^{(1)}(c,\xi) + B_{q,n} R_{q,n}^{(2)}(c,\xi) = 0 \quad (25)
\]
\[
\sum_{q,n} \left[ \left( D_{q,n} R_{q,n}^{(1)}(c,\xi) + P_{q,n} R_{q,n}^{(2)}(c,\xi) \right) S_{q,n}(c,\eta) \right] + \frac{1}{j\kappa h} \sum_{q,n} \left[ \left( D_{q,n} R_{q,n}^{(1)}(c,\xi) + P_{q,n} R_{q,n}^{(2)}(c,\xi) \right) S_{q,n}(c,\eta) \right] = 0.
\]

III. NUMERICAL RESULTS

In the limit \( \xi \to \infty \), since \( c\xi \to k\rho \) with \( \rho \) being the radial cylindrical coordinate, using asymptotic expressions, the radial Mathieu function of the fourth kind and its first derivative with respect to argument can be written as:

\[
\lim_{\xi \to \infty} R_{q,n}^{(1)}(c,\xi) \approx \frac{j^n}{\sqrt{k\rho}} e^{-k\rho},
\]
\[ \lim_{\xi \to \infty} R^{(iv)}(c, \xi) \approx \frac{j^{n-1} \rho}{k \rho} \left( \frac{1}{k \rho} \right) e^{-\beta \rho}. \] (31)

Using (31) and the fact that in the limit \[ \xi \to \infty, \]
\[ kh \to kF \cosh u \approx k \rho, \]
we can write expressions for the scattered electric field components in the far zone as:

\[ E^e_z = \frac{j}{k \rho} e^{-\beta \rho} \left[ \sum_{n=0}^{n} f^n B_{m} S_{m}(c, \cos \phi) + \sum_{n=1}^{\infty} j^n B_{m} S_{m}(c, \cos \phi) \right], \] (32)

\[ E^o_z = \frac{j}{k \rho} e^{-\beta \rho} \left[ \sum_{n=0}^{n} j^n C_{m} S_{m}(c, \cos \phi) + \sum_{n=1}^{\infty} j^n C_{m} S_{m}(c, \cos \phi) \right], \] (33)

and the scattered magnetic field components as

\[ H^i_z = E^e_z / Z \quad \text{and} \quad H^o_z = -E^o_z / Z. \]

The bistatic scattering cross section is defined as:

\[ \sigma = \lim_{\rho \to \infty} 2 \pi \rho \left\{ \frac{\text{Re}[E_z \times \hat{H} \cdot \hat{p}]}{\text{Re}[E_z \times \hat{H} \cdot \hat{p}]} \right\}, \] (34)

with \( \text{Re}[w] \) denoting the real part of the complex number \( w \), the asterisk denoting the complex conjugate, and \( \hat{p} \) denoting the unit vector in the increasing radial direction. Substituting for the far zone scattered fields in (34), and recalling that the incident field is of unit magnitude for each angle being a maximum for \( \upsilon=0^\circ \) (PEC), and the differences of two values of \( \upsilon \) becoming a maximum at \( \phi=180^\circ \), corresponding to forward scattering. The normalized bistatic widths are compared with the corresponding values obtained for a conventional coated PEMC elliptic cylinder in [22], presented by circles, and are in very good agreement, validating the calculations for an elliptic cylinder in general.

Figure 3 (a) is similar to Fig. 2, except that the chiral admittance is taken to be \( \zeta_c = 0.002. \) This figure shows the effect of both PEMC and chiral coating on the bistatic of coated elliptic cylinder. There is a drop in the bistatic width by approximately of 50% at \( \upsilon=0^\circ \) and \( \upsilon=90^\circ \), and no change in the location of the maximum values for other values of \( \upsilon \). It is worth mentioning that the value of the bistatic width for \( \upsilon=75^\circ \) and \( 90^\circ \) are the same at \( \phi=180^\circ. \) Figure 3 (b) is similar to Fig. 3 (a), except \( \zeta_c \) is reduced to 0.0015. It can be seen that the bistatic width is higher and the maximum is back at \( \upsilon=0^\circ \) and the minimum at \( \upsilon=90^\circ. \) The variation of the bistatic widths is due to the presence of the cross-polarized fields of
PEMC and chiral materials.

Figure 4 is similar to Fig. 3, except that the incident angle is 90°. Figure 4 (a) shows that the magnitude of the bistatic width has maxima at $\nu=90^\circ$ and minima at $\nu=0^\circ$ and $\varphi=90^\circ$, while the opposite is happening at the scattering angle of $\varphi=270^\circ$. Figure 4 (b) is for the case of $\zeta_c=0.0015$.

![Figure 4](image)

Fig. 4. Normalized bistatic width versus scattering angle for PEMC elliptic cylinders of different admittances coated with $\varepsilon_{rc}=2.5$, $\mu_{rc}=1.0$, $\phi_i=90^\circ$: (a) $\zeta_c=0.002$, and (b) $\zeta_c=0.0015$.

It can be seen that Fig. 5 is similar to 4, except by increasing $\varepsilon_{rc}$ from 2.5 to 3.0. Figure 6 shows the bistatic widths versus the major axis of the dielectric coating for $\nu=45^\circ$ and $75^\circ$ and $\zeta_c=0.002$ and 0.0025. The presence of chiral material effects the magnitude but not the pattern of the bistatic width. More results and analysis on chiral coated or PEMC conventional coated elliptic cylinder can be found in [22,24].

![Figure 5](image)

Fig. 5. Normalized bistatic width versus scattering angle for PEMC elliptic cylinders of different admittances coated with $\varepsilon_{rc}=3.0$, $\mu_{rc}=1.0$, $\phi_i=90^\circ$ and $\zeta_c=0.002$. 
ε elliptic cylinders of different admittances coated with 

Fig. 6. Normalized bistatic width versus ka1 for PEMC 

(a)

(b)

Fig. 6. Normalized bistatic width versus ka1 for PEMC 

electric cylinders of different admittances coated with 

(a) υ=45°, and 

(b) υ=75°.

IV. CONCLUSION

Analytic solution has been obtained to the problem 
of scattering from a chirally coated PEMC elliptic 
cylinder, when it is excited by a uniform plane wave. The 
solution is general since it also can provide the solution 
to the scattering by PEMC circular or strip chiral coated 
geometries. The results obtained show that the 
admittances as well as the constitutive parameters of the 
chiral coating material can be used to control (enhancing 
or reducing) the scattering width of a coated PEMC 
electric cylinder. Thus, the solution provides the designer 
with two degree of freedom to control the bistatic width. 
The new results obtained in this paper are important, 
since they can be used to validate similar results obtained 
using other methods, and provide an insight into how the 
changing of various parameters associated with a 
chirally coated PEMC elliptic cylinder changes the 
scattering widths that could be obtained from it.

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