The Computational Expressions of Spheroidal Eigenvalues

Wan-xian Wang
Space Astronomy Laboratory, University of Florida
Gainesville, FL 32609

ABSTRACT

The higher order terms of eigenvalues in spheroidal differential equation are developed by using power-series expansion and asymptotic ones for both prolate and oblate wave functions, these important multipole expansions greatly facilitate and improve the computations of the electromagnetic scattering by different kinds of spheroids with various size parameters, refractive indices, and aspect ratios.

I. INTRODUCTION

The prolate and oblate spheroidal eigenvalues $\lambda_{mn}$ are usually calculated following Bouwkamp's method\(^1\) while restricted to the case of small value of size parameter $c$

\[c = \kappa (a^2-b^2)^{1/4}\]

where $\kappa$ is wave number, $a$ is semi-major axis, and $b$ is semi-minor axis) and large number $n$. For large value of $c$ (say $> 15$) and/or small number $n$, the asymptotic expansions must be employed; J. Meixner\(^2\),\(^3\) had performed the asymptotic developments of prolate and oblate spheroidal eigenvalues up to $c^{-5}$, respectively. However, for moderate value of $c$ and the intermediate number $n$, there appears a gap between Bouwkamp's and the asymptotic expansions because of the orders of the included terms being not high enough for these expansions.

The author has pushed the power-series expansion forward to $c^{16}$ term, and the prolate and oblate asymptotic developments till $c^{-6}$, respectively. Thus the correct arrangement of the eigenvalues $\lambda_{mn}$ from small value through large value of $c$ and from small number through large number $n$ is formed in increasing order by correspondingly selecting one of these two expansions. The developments of the analytical expressions of the spheroidal eigenvalues, together with further improvements on calculating the spheroidal radial functions, have made it feasible to compute the scattering coefficients for different kinds of spheroids within very wide range.

112
II. POWER-SERIES EXPANSION OF THE SPHEROIDAL EIGENVALUES

The angular differential equation of the spheroidal wave functions can be written in the form

\[
(1 - \eta^2) \frac{d^2S_{mn}}{d\eta^2} - 2\eta \frac{dS_{mn}}{d\eta} + (\lambda_{mn} - c^2\eta^2 - \frac{m^2}{1 - \eta^2})S_{mn} = 0
\]

where \( \eta \) is angular coordinate in the spheroidal system, \(-1 \leq \eta \leq 1\);

\( S_{mn} \) are the spheroidal angle functions of order \( m \) and degree \( n \);

\( \lambda_{mn} \) are the eigenvalues of the spheroidal differential equation;

\( m \) and \( n \) are positive integers with \( n \geq m \).

This equation is for prolate spheroidal wave functions. By replacing \( c \) by \(-ic\) in Eq. (1) we would have the equation for oblate one. For small value of \( c \), the following expression of the eigenvalues \( \lambda_{mn} \) in the form of continued fraction can be obtained by using three-term recursion relation of the expansion coefficients, \( c_r^{mn}(c) \), of the spheroidal angle functions \( S_{mn} \) with respect to the associated Legendre functions \( P_{m+r}^m \) (where \( r \) is the summation index); \(^4\)

\[
\lambda_{mn} = \gamma_{n-m}^m - \frac{\beta_{n-m}}{\gamma_{n-m-2}^m - \lambda_{mn}^m} \gamma_{n-m-2}^m - \lambda_{mn}^m - \frac{\beta_{n-m+2}}{\gamma_{n-m+2}^m - \lambda_{mn}^m} \gamma_{n-m+2}^m - \lambda_{mn}^m - \cdots
\]

where \( \gamma_r^m = (m+r)(m+r+1) + \frac{1}{2}c^2 \left[ 1 - \frac{4m^2-1}{(2m+2r-1)(2m+2r+3)} \right] \quad (r \geq 0) \)

\( \beta_r^m = \frac{r(r+1)(2m+r)(2m+r-1)c^4}{(2m+2r-1)^2(2m+2r-3)(2m+2r+1)} \quad (r \geq 2) \)

Substituting in Eq. (2) the power-series expansion

\[
\lambda_{mn} = \sum_{k=0}^{\infty} \gamma_{2k}^m c^{2k}
\]

and then developing the continued fraction by raising consecutively each partial denominator up to the associated numerator with the use of binomial expansion, we can find the coefficients \( \gamma_{2k}^m \) by equating the power of \( c^{2k} \). The coefficients in the book by C. Flammer\(^5\) were given till \( \gamma_{10}^m \); it might be remarked that he obtained the coefficient
\( \lambda_{mn} \) as given below, but apart from a sign error in the third term of the second part (the numerator \( 6n-25 \) instead of the correct \( 6n+25 \)). In most cases, the coefficients up to \( \lambda_{12} \) are often used, the coefficients \( \lambda_{14}^{mn} \) and \( \lambda_{16}^{mn} \) are just applied to some certain size parameters of moderate value of \( c \) and intermediate numbers \( n \). To save space the coefficients derived by author are not fully listed; \( \lambda_{14}^{mn} \) and \( \lambda_{16}^{mn} \) can be found from author's paper. 

The power-series expansion of the spheroidal eigenvalues \( \lambda_{mn} \) is as follows:

\[
\lambda_{mn} = \sum_{k=0}^{\infty} \lambda_{2k}^{mn} c^{2k}
\]

where

\[
\lambda_{0}^{mn} = n(n+1)
\]

\[
\lambda_{2}^{mn} = \frac{1}{2} \left[ 1 - \frac{4m^{2}-1}{(2n-1)(2n+3)} \right]
\]

\[
\lambda_{4}^{mn} = \frac{\beta_{0}}{2(2n-1)} - \frac{\beta_{2}}{2(2n+3)}
\]

\[
\lambda_{6}^{mn} = -\frac{\beta_{0}(4m^{2}-1)}{4(2n-1)^{2}} \left[ \frac{4}{(2n-5)(2n-1)(2n+3)} \right] + \frac{\beta_{2}(4m^{2}-1)}{4(2n+3)^{2}} \left[ \frac{4}{(2n+7)(2n+3)(2n-1)} \right]
\]

\[
\lambda_{8}^{mn} = -\frac{\beta_{0}(4m^{2}-1)}{4(2n-1)^{2}} \left[ \frac{\lambda_{4}^{mn}}{4(2n-1)(2n-3)} \right] - \frac{\beta_{2}(4m^{2}-1)}{4(2n+3)^{2}} \left[ \frac{\lambda_{4}^{mn}}{(2n+7)(2n+3)(2n-1)} \right] + \frac{\beta_{4}}{(2n+5)}
\]

\[
\lambda_{10}^{mn} = -\frac{\beta_{0}(4m^{2}-1)}{4(2n-1)^{2}} \left[ \frac{\lambda_{6}^{mn}}{(2n-5)(2n-1)^{2}(2n+3)} \right] + \frac{\lambda_{8}^{mn}}{(2n-5)(2n-1)(2n+3)(2n+5)(2n+3)^{3}}
\]

\[
+ \frac{\beta_{2}(6n-19)}{2(2n-9)(2n-5)(2n-3)(2n-1)^{2}(2n+3)}
\]

\[
- \frac{\beta_{2}(4m^{2}-1)}{4(2n+3)^{2}} \left[ \frac{\lambda_{6}^{mn}}{(2n+7)(2n+3)^{2}(2n-1)} \right] - \frac{\lambda_{8}^{mn}}{(2n+7)(2n+3)^{5}(2n-1)^{3}}
\]

\[
- \frac{\beta_{4}(6n+25)}{2(2n+11)(2n+7)(2n+5)(2n+3)^{2}(2n-1)}
\]

\[\text{114}\]
\[
\lambda_{nm}^{12} = \frac{\beta_0(4m^2-1)}{4(2n+1)^2} \left\{ \frac{\lambda_8^{mn}}{(4m^2-1)} - \frac{(\lambda_4^{mn})^2}{2(4m^2-1)(2n+1)} - \frac{4\epsilon_6^{mn}}{(2n-5)(2n+1)^2(2n+3)} + \frac{12(4m^2-1)\lambda_6^{mn}}{(2n-5)^2(2n-1)^4(2n+3)^2} \right.
\]
\[\left. - \frac{32(4m^2-1)^3}{(2n-5)^4(2n-1)^7(2n+3)^4} - \frac{\beta_2}{(4m^2-1)(2n-3)} + \frac{\beta_6}{96(2n-5)(2n-3)} \right\} \left\{ \frac{\lambda_8^{mn}}{(4m^2-1)} + \frac{(\lambda_4^{mn})^2}{2(4m^2-1)(2n+3)} - \frac{4\epsilon_6^{mn}}{(2n+7)(2n+3)^2} + \frac{12(4m^2-1)\lambda_6^{mn}}{(2n+7)^2(2n+3)^4(2n+1)^2} \right.
\]
\[\left. + \frac{32(4m^2-1)^3}{(2n+7)^4(2n+3)^7(2n+1)^4} + \frac{\beta_4}{(4m^2-1)(2n+5)} + \frac{\beta_6}{96(2n+7)(2n+5)} \right\} \]  

(4-7)

where \( \beta_4 = \beta_{n-m/4}/c^4 \), \( \beta_2 = \beta_{n-m/2}/c^4 \), \( \beta_0 = \beta_{n-m}/c^4 \), \( \beta_6 = \beta_{n-m+6}/c^4 \).

and \( d_1 = (2n-1)^2 + 2(2n-1)(2n-9) + 3(2n-9)^2 \)

\( d_2 = (2n+3)^2 + 2(2n+3)(2n+11) + 3(2n+11)^2 \)

The power-series expansion of the oblate eigenvalues \( \lambda_{mn} \) is obtained from Eq. (4) by simply replacing \( c^2 \) by \(-c^2\).

III. ASYMPTOTIC DEVELOPMENT OF THE PROLATE SPHEROIDAL EIGENVALUES

Let us set

\[
S_{mn} = (1 - \eta^2)^{\frac{m}{2}} u_{mn}
\]  

(5)

substitute this expression in Eq. (1), and make the transformation

\[
\eta = (2c)^{-\frac{1}{4}} x
\]  

(6)

there results

\[
(1 - \frac{x^2}{2c}) \frac{d^2u_{mn}}{dx^2} - \frac{(m+1)}{c} \frac{du_{mn}}{dx} + (K - \frac{x^2}{a}) u_{mn} = 0
\]  

(7)

115
where constant \( K = \frac{\lambda_{mn}}{2c} - m(m+1) \) \hspace{1cm} (7-1)

First of all, developing \( u_{mn} \) and \( K \) in the asymptotic forms of size parameter \( c \):

\[ u_{mn} = u_0 + u_1 c^{-1} + u_2 c^{-2} + u_3 c^{-3} + \ldots + u_k c^{-k} \] \hspace{1cm} (8-1)

\[ K = a_0 + a_1 c^{-1} + a_2 c^{-2} + a_3 c^{-3} + \ldots + a_k c^{-k} \] \hspace{1cm} (8-2)

and substituting them in Eq. (7), we can determine the terms \( u_0, u_1, u_2, \ldots \) of eigenfunctions \( u_{mn} \) by a series of differential equations of the second order, namely —

\[ u_0'' + (a_0 - \frac{x^2}{4})u_0 = 0 \] \hspace{1cm} (9-1)

\[ u_1'' + (a_0 - \frac{x^2}{4})u_1 = -a_1 u_0 + \frac{1}{2}[x^2 u_0'' + 2(m+1) xu_0'] \] \hspace{1cm} (9-2)

\[ u_2'' + (a_0 - \frac{x^2}{4})u_2 = -a_1 u_1 - a_2 u_0 + \frac{1}{2}[x^2 u_1'' + 2(m+1) xu_1'] \] \hspace{1cm} (9-3)

\[ \ldots \ldots \]

\[ u_k'' + (a_0 - \frac{x^2}{4})u_k = -a_1 u_{k-1} - a_2 u_{k-2} - \ldots - a_k u_0 + \frac{1}{2}[x^2 u_{k-1}'' + 2(m+1) xu_{k-1}'] \] \hspace{1cm} (9-k+1)

Let

\[ a_0 = p + \frac{1}{2} \] \hspace{1cm} (10)

where \( p \) is a positive integer or zero. The solution of Eq. (9-1) is

\[ u_0 = D_p(x) \] \hspace{1cm} (11)

where \( D_p(x) \) is parabolic cylinder function. For simplicity, we just denote it by \( D_p \).

In the first approximation of \( K \), we have

\[ K = a_0 = p + \frac{1}{2} \] \hspace{1cm} (12)

It implies that

\[ \lambda_{mn} = (2p + 1)c \] \hspace{1cm} (13)

while \( c \to \infty \)

According to the asymptotic property of the eigenvalues \( \lambda_{mn} \), we find that

\[ \lambda_{mn} = [2(n - m) + 1]c \] \hspace{1cm} (14)

while \( c \to \infty \)

The foregoing suggests that

\[ p = n - m \] \hspace{1cm} (15)

Next we define the operator 8
\[ \nabla D_p = x^2 D_p'' + 2(m+1)x D_p' \]

Utilizing the recursion relation of the parabolic cylinder functions:
\[ D_p' + \frac{x}{2} D_p - pD_{p-1} = 0 \]
we obtain
\[ \nabla D_p = C_1 D_{p+4} + C_2 D_{p+2} + C_3 D_p + C_4 D_{p-2} + C_5 D_{p-4} \]
where
\[ C_1 = \frac{1}{4} \]
\[ C_2 = -m \]
\[ C_3 = -\frac{2p^2 + 2p + 3}{4} - m \]
\[ C_4 = \frac{|p|_2}{2} \]
\[ C_5 = \frac{|p|_4}{4} \]
with \(|p|_2 = p(p-1)\), and \(|p|_4 = p(p-1)(p-2)(p-3)\).

Again, we define the operator
\[ \Delta u_r = u_r'' + \left( p + \frac{1}{2} - \frac{x^2}{4} \right) u_r \]
where \( r = 1, 2, \ldots k \).

Remembering that
\[ \Delta D_p = 0 \]
we find that \( D_p \) term will not appear in the solution \( u_r \) of Eqs. (9-2), (9-3), \ldots.

Furthermore we set
\[ u_r = \sum_{\lambda \neq 0} A_{2\lambda} D_{p+2\lambda} \]
then
\[ \Delta u_r = \sum_{\lambda \neq 0} (-2\lambda) A_{2\lambda} D_{p+2\lambda} = \sum_{\lambda \neq 0} B_{2\lambda} D_{p+2\lambda} \]
where we have expressed the right-hand sides of Eqs. (9-2), (9-3), \ldots in the forms
of \( \sum_{\lambda \neq 0} B_{2\lambda} D_{p+2\lambda} \).

Hence the coefficients \( A_{2\lambda} \) are found to be
\[ A_{2l} = -\frac{B_{2l}}{2l} \]  

(23)

Now we can solve Eqs. (9-2), (9-3), ... .

From Eq. (9-2), we get

\[ \alpha_1 = \frac{C_3 \cdot p}{2} - \frac{2p^2 + 2p + 3}{8} - \frac{m}{2} \]  

(24)

By assuming

\[ q = 2p + 1 \]  

(25)

we have

\[ \alpha_1 = -\frac{q^2 + 5 + 8m}{16} \]  

(26)

The term \( u_1 \) of the eigenfunctions \( u_{mn} \) is found in the form

\[ u_1 = \frac{1}{2} \left[ f_1 D_{p+4} + f_2 D_{p+2} + f_3 D_{p-2} + f_4 D_{p-4} \right] \]  

(27)

where

\[ f_1 = -\frac{1}{16} \]  

(27-1)

\[ f_2 = \frac{m}{2} \]  

(27-2)

\[ f_3 = \frac{m}{2} |p| \]  

(27-3)

\[ f_4 = \frac{1}{16} |p| \]  

(27-4)

Similarly we can obtain the other coefficients \( \alpha_r \) of the eigenvalues \( \lambda_{mn} \) and the terms \( u_r \) of the eigenfunctions \( u_{mn} \) by the successive substitutions. The expression of terms \( u_r \) will be lengthy and lengthy while \( r \) increases; therefore I only list the coefficients \( \alpha_r \).

\[ \alpha_2 = -\frac{q(q^2 + 11 - 32m^2)}{2^7} \]  

(28)

\[ \alpha_3 = -\frac{5(q^4 + 26q^2 + 21) - 384m^2(q^2 + 1)}{2^{11}} \]  

(29)

\[ \alpha_4 = -\frac{(33q^8 + 1594q^3 + 5621q) - 128m^2(37q^3 + 167q) + 2048m^4q}{2^{15}} \]  

(30)

\[ \alpha_5 = -\frac{(63q^6 + 4940q^4 + 43327q^2 + 22470) - 128m^2(115q^4 + 1310q^2 + 735) + 24576m^4(q^2 + 1)}{2^{17}} \]  

(31)
\( \alpha_6 = - \left[ (527q^7 + 61529q^5 + 1043961q^3 + 2241599q) - 32m^2 (5739q^5 + 127550q^3 + 298951q) + 2048m^4 (355q^3 + 1505q) - 65536m^6 q \right] 2^{-21} \)  
\( \alpha_7 = - \left[ (9387q^8 + 1536556q^6 + 43711178q^4 + 230937084q^2 + 93110115) - 1536m^2 (2989q^6 + 112020q^4 + 648461q^2 + 270690) + 196608m^4 (175q^4 + 1814q^2 + 939) - 12582912m^6 (q^2 + 1) \right] 2^{-26} \)  

Therefore the eigenvalues \( \lambda_{mn} \) are in the form:  
\( \lambda_{mn} = q c - \frac{1}{8} (q^2 + 5 - 8m^2) + \sum_{r=2}^{7} 2a_r c^{-r+1} + O(c^{-7}) \)  

Correspondingly the eigenfunctions \( u_{mn} \) are in the form:  
\( u_{mn} = \sum_{r=0}^{6} u_r c^{-r} + O(c^{-7}) \)  

As E.L.Ince said: "if anyone had the courage to push the development on a stage or two further he would greatly enhance the value of an important expansion. But any reader who attempts to verify the results given above will realize that the work involved would be tremendous".  

The expression of the eigenvalues \( \lambda_{mn} \) of the prolate spheroidal differential equation can be converted to that of the eigenvalue \( \Lambda \) of the Mathieu differential equation, as follows:  
\[
\Lambda = -2h^2 + 2qh - \frac{1}{8} (q^2 + 1) - (q^3 + 3q) 2^{-7} h^{-1} - (5q^4 + 34q^2 + 9) 2^{-12} h^{-2} - (33q^5 + 410q^3 + 405q) 2^{-17} h^{-3} - (63q^6 + 1260q^4 + 2943q^2 + 486) 2^{-20} h^{-4} - (527q^7 + 15617q^5 + 69001q^3 + 41607q) 2^{-25} h^{-5} - (9387q^8 + 388780q^6 + 2845898q^4 + 4021884q^2 + 506979) 2^{-31} h^{-6} + O(h^{-7})
\]  

where  
\[
h = \frac{c}{2}
\]  
\[
\Lambda = \lambda_{mn} + \frac{1}{4} - \frac{1}{2} c^2
\]  

herein the author has developed one more high order term \( h^{-6} \).

IV. ASYMPTOTIC DEVELOPMENT OF THE OBLATE SPHEROIDAL EIGENVALUES

The oblate spheroidal differential equation for angle functions \( S_{mn} \) is expressed as  
\[
(1 - \eta^2) \frac{d^2 S_{mn}}{d\eta^2} - 2n \frac{dS_{mn}}{d\eta} + (\lambda_{mn} + c^2 \eta^2 - \frac{m^2}{1 - \eta^2}) S_{mn} = 0
\]  

(37)
Referring to C. Flammer's book, and using three-term recursion relation of the expansion coefficients, \( \Lambda_{mn} \), of the oblate spheroidal angle functions \( S_{mn} \) with respect to the Laguerre functions \( L_{\nu+s}^{(m)} \) (where \( \nu = \frac{1}{2} (n-m) \) if \( n-m \) is even, and \( \nu = \frac{1}{2} (n-m-1) \) if \( n-m \) is odd, and \( s \) is the summation index), we have the following expression of the eigenvalues \( \Lambda_{mn} \) in the form of transcendental equation:

\[
\Lambda_{mn} = \frac{Q_0^2}{\Lambda_{mn} + P_{-1}} - \frac{Q_{-1}^2}{\Lambda_{mn} + P_{-2}} + \frac{Q_1^2}{\Lambda_{mn} + P_1} - \frac{Q_2^2}{\Lambda_{mn} + P_2} - \ldots \tag{38}
\]

where

\[
Q_s = (s+\nu)(s+\nu+m) \tag{38-1}
\]

\[
P_s = 2s(2s+m+1-2c+s) \tag{38-2}
\]

The eigenvalues \( \lambda_{mn} \) are related to the eigenvalues \( \Lambda_{mn} \) by

\[
\lambda_{mn} = -c^2 + 2qc - \frac{1}{2}(q^2+1-m^2) + \Lambda_{mn} \tag{39}
\]

where

\[
q = n + 1 \quad \text{while } (n-m) \text{ even} \tag{39-1}
\]

\[
q = n \quad \text{while } (n-m) \text{ odd} \tag{39-2}
\]

Substituting in Eq. (38) the inverse power-series with respect to size parameter \( c \):

\[
\Lambda_{mn} = \sum_{i=1}^{\infty} \frac{1}{i!} \lambda_{mn} c^{-i} \tag{40}
\]

and then expanding the continued fraction by doing same procedure as power-series expansion discussed in Section II, we can get the coefficients \( \lambda_{i} \). The eigenvalues \( \lambda_{mn} \) of the oblate spheroidal differential equation would be:

\[
\lambda_{mn} = -c^2 + 2qc - \frac{1}{2}(q^2+1-m^2) - q(q^2+1-m^2)z^{-3}c^{-1} - [(5q^8+10q^6+12q^4+10q^2+1)+\frac{1}{2}(3q^2+1)+m^4]z^{-2}c^{-2} - q[(33q^6+114q^4+37+2m^2)(23q^2+25)+13m^2]z^{-3}c^{-3} - [(63q^6+340q^4+239q^2+14)+10m^2((10q^4+23q^2+3)+3m^4)(13q^2+6)-2m^2]z^{-10}c^{-10} - q[(527q^8+4139q^6+5221q^4+1009)-m^2(939q^6+3750q^4+1591)+5m^4(93q^2+127)-53m^2]z^{-13}c^{-5} - [(9387q^8+101836q^6+205898q^4+86940q^2+3747)-12m^2((1547q^6+9575q^4+8657q^2+701)+6m^4(1855q^4+5078q^2+939)+12m^6(167q^2+85)+51m^8)z^{-17}c^{-6} + 0c^{-7} \tag{41}
\]
V. DISCUSSION

In order to obtain the accurate spheroidal eigenvalues, we should substitute the expression of spheroidal eigenvalues with higher order terms for Bouwkamp's or asymptotic expansion, as the initial values, into Eq. (2). Since the terms in the multipole expansion have been developed in such a high order that the initial eigenvalues will be immediately bound within the convergence circles. Therefore, the final eigenvalues $\lambda_{mn}$ can be easily reached by iterated procedures at very fast convergence rates.

The spheroidal eigenvalues $\lambda_{mn}$, the spheroidal angular functions $S_{mn}$, and the spheroidal radial functions $R_{mn}$, together with the boundary conditions matching, make the computational electromagnetic scattering problems solvable.

REFERENCES


